An Atlas of Differential Topology

From Fundamentals to Stokes

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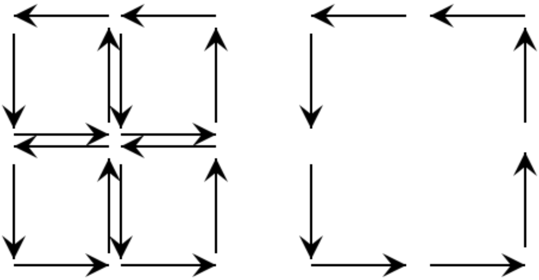


Goal

What is integration?

What is the little $\mathrm{d}x$ **at the end of the integral sign?**

Evolution of Calculus

Calculus I/II — So many rules... and a lot of computation. Integration is FToC

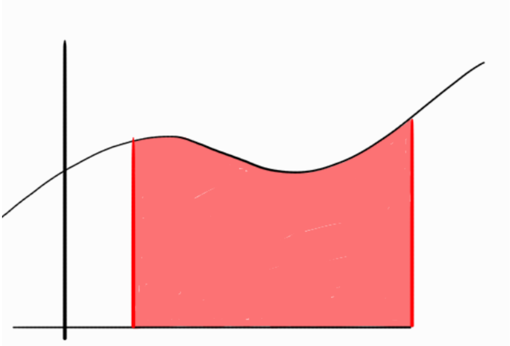
Calculus III — Vector fields, grad/curl/div and all that... I can't visualize in 7 dimensions, can you? Also, what's a determinant?

Linear Algebra — What do linear transformations of vector spaces have to do calculus? I miss integration.

Differential Topology — Wait, it's been linear algebra since the beginning?

Integration: A First Encounter

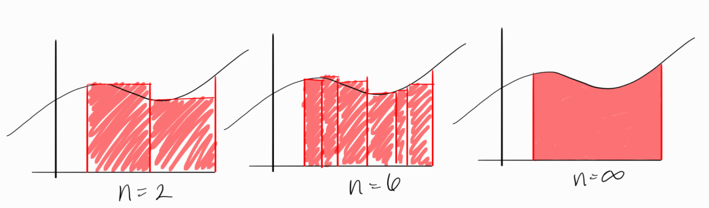
f is some continuous function:



How do we compute the area of the colored region?

Riemann Sums!

Riemann Sums



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

where $\{x_k\}_{k=0}^{n}$ partitions $[a, b]$
and $x_k^* \in [x_{k-1}, x_k]$

But as if by magic...

Let f measure the *rate of change* of another function F.

Net change in F from a to b can be expressed as the sum of all the tiny changes of F:

Theorem (Fundamental Theorem of Calculus)

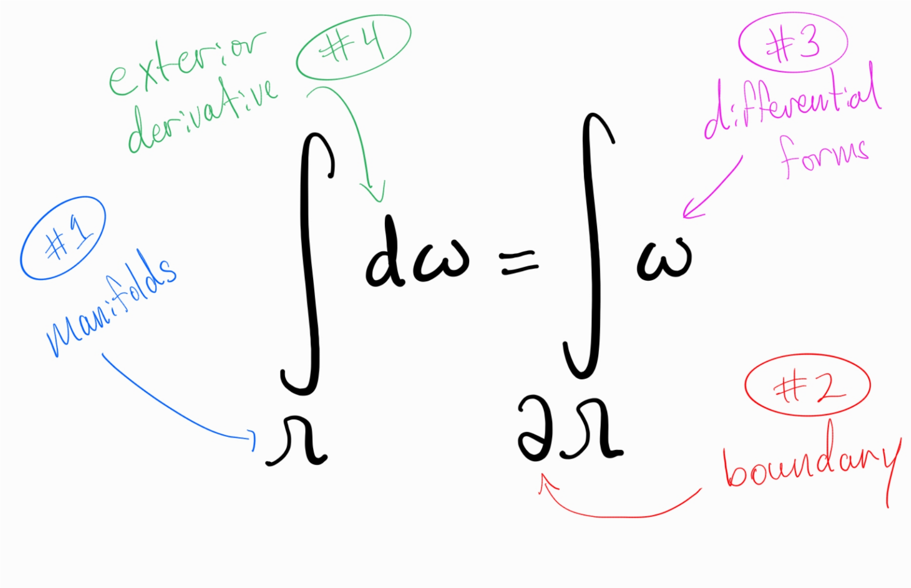
Let $F : [a, b] \to \mathbb{R}$ be continuously differentiable with derivative f. Then

$$\int_a^b f \, \mathrm{d}x = F(b) - F(a)$$

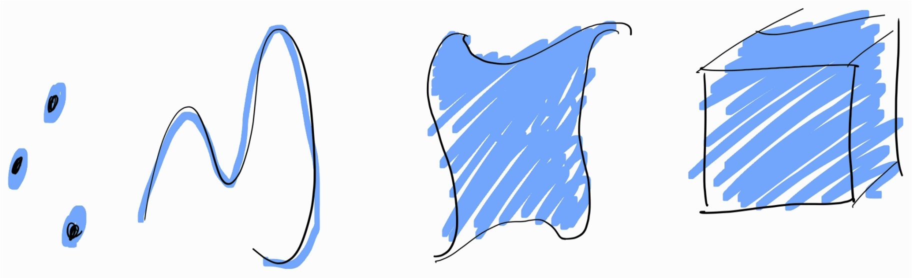
This is actually a special case of a much more beautiful result:

$$\int_{\Omega} \mathrm{d}\omega = \int_{\partial\Omega} \omega$$

The Cast of our Show



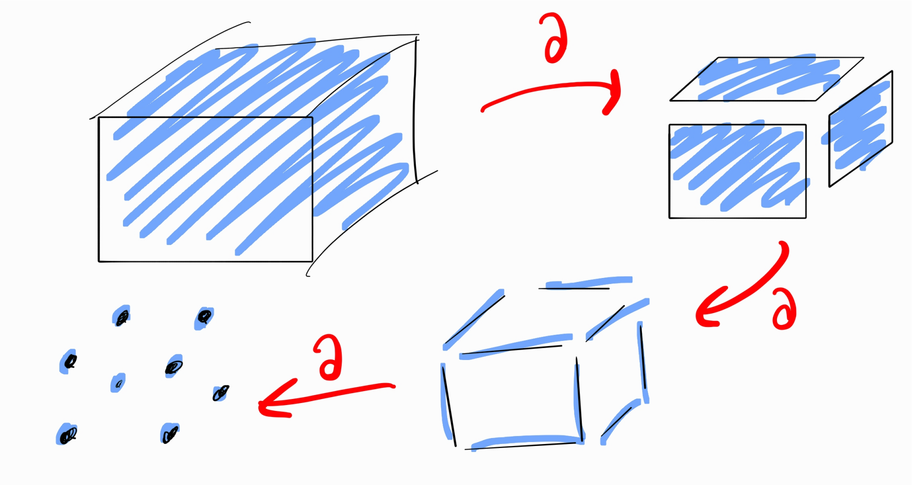
Manifolds



From left to right: 0-dimensional through 3-dimensional manifolds.

Example: the unit disk $\mathbb{D}^2$

Boundary

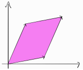
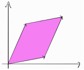


Example: $\partial\mathbb{D}^2 = S^1$, the unit sphere

Differential forms

From linear algebra, the **determinant** is a function that gives the signed geometric volume from a square matrix.

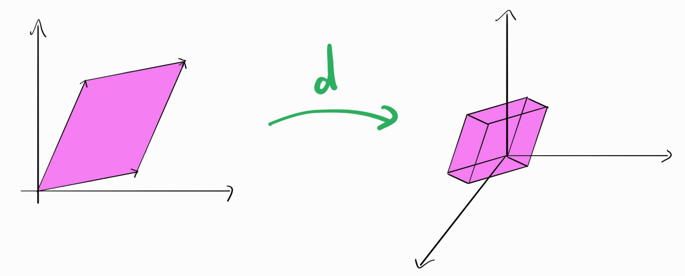
Differential *k*-forms are *functions* that give signed k-volumes of $n \times k$ (non-square) matricies: the signed volume of an k-parallelepiped existing in $\mathbb{R}^n$.

$\mathrm{d}x \wedge \mathrm{d}y$ () $= \mathrm{Area}$ ()

Example: the 1-form $\omega = y\,\mathrm{d}x - x\,\mathrm{d}y$

The Exterior Derivative

Generalization of derivatives to forms.



A procedure to generate an appropriate $(k+1)$-form from a k-form.

Example: the 2-form $\mathrm{d}\omega = \mathrm{d}(y\,\mathrm{d}x - x\,\mathrm{d}y) = -2\,\mathrm{d}x \wedge \mathrm{d}y$

Two integrals

$$\int_{\mathbb{D}^2} \mathrm{d}\omega = \int_{\mathbb{D}^2} -2\,\mathrm{d}x \wedge \mathrm{d}y$$
$$= -2 \int_{\mathbb{D}^2} \mathrm{d}x \wedge \mathrm{d}y = -2 \cdot \mathrm{Vol}_2\left(\mathbb{D}^2\right) = -2\pi$$

$$\int_{\partial\mathbb{D}^2} \omega = \int_{S^1} \left(y\,\mathrm{d}x - x\,\mathrm{d}y\right)$$
$$= \int_{\gamma([0,2\pi])} P_{\langle \cos t, \sin t\rangle}(y\,\mathrm{d}x - x\,\mathrm{d}y)$$
$$= \int_0^{2\pi} ((-\sin t)(\sin t) - (\cos t)(\cos t))\,\mathrm{d}t$$
$$= -\int_0^{2\pi} \mathrm{d}t = -\mathrm{Vol}_1\left([0, 2\pi]\right) = -2\pi$$

Stokes' Theorem on Manifolds

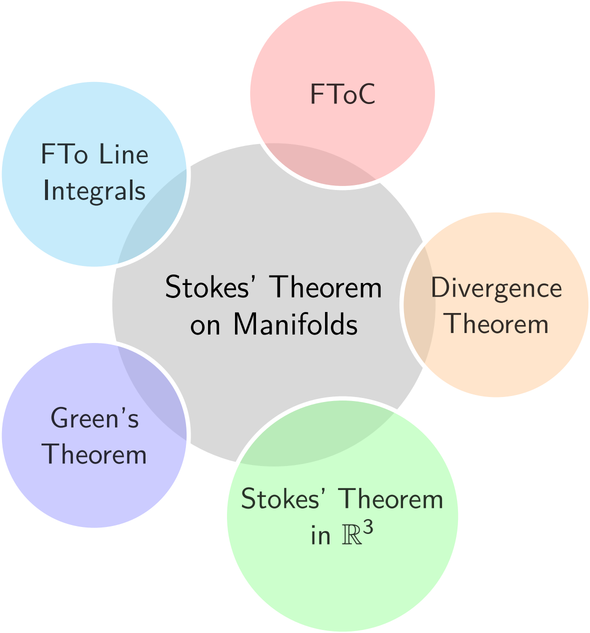
Theorem

Let ω be a differential form, and let Ω be an orientable manifold with boundary. Then

$$\int_{\Omega} \mathrm{d}\omega = \int_{\partial\Omega} \omega. \tag{1}$$

The integral over the *boundary* is the integral of the *change* over the region.

FToC	Green's Theorem	Divergence Theorem
$\int_{[a,b]} f' \, \mathrm{d}x = \int_{\partial[a,b]} f$	$\int_S \mathrm{d}W_{\vec{F}} = \int_{\partial S} W_{\vec{F}}$	$\int_X \mathrm{d}\varphi = \sum_i \int_{S_i} \varphi$


FToC
FTo Line Integrals
Stokes' Theorem on Manifolds
Divergence Theorem
Green's Theorem
Stokes' Theorem in $\mathbb{R}^3$

THANK YOU!

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An Atlas of Differential Topology

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