

subject: Linear Algebra

date:

topic: 5.2 (Down with) Determinants

keywords: Determinants, Cofactor Matrices, Invertible Matrix Theorem

Test 3	Monday 5/1 11:30 am	(8:25 class)
Final	5/3 8 am	(9:30 class)

A function

$\det : M_{n\times n} \to \mathbb{R}$

i.e. for $A \in M_{n\times n}$, $\det A \in \mathbb{R}$ → $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}$; $\det A = ad - bc$

Ex $A = \begin{bmatrix} 7 & 2 \\ 4 & -3 \end{bmatrix}$; $\det A = 7(-3) - (2)(4) = -29$

A geometric interpretation

If you hit a unit square w/ A, area will scale by -29 (negative $\stackrel{?}{=}$ rotation)

For $A \in M_{n\times n}$, $[a_{ij}] = A$

row, column

$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}$

ij cofactor matrix

A w/o i-th row, j-th column

$$A = \begin{array}{c} \begin{matrix} a_{11} & a_{12} & a_{13} \end{matrix} \\ \begin{bmatrix} 7 & 2 & 4 \\ 0 & -1 & 5 \\ 0 & 4 & 0 \end{bmatrix} \end{array}$$

$\det A = 7 \det \begin{bmatrix} -1 & 5 \\ 4 & 0 \end{bmatrix} - 2 \det \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} + 4 \det \begin{bmatrix} 0 & -1 \\ 0 & 4 \end{bmatrix}$ → cofactor expansion across 1st row

$\begin{bmatrix} 7 & 2 & 4 \\ 0 & -1 & 5 \\ 0 & 4 & 0 \end{bmatrix}$ (1,1) cofactor

$\begin{bmatrix} 7 & 2 & 4 \\ 0 & -1 & 5 \\ 0 & 4 & 0 \end{bmatrix}$ (1,2) cofactor

$\begin{bmatrix} 7 & 2 & 4 \\ 0 & -1 & 5 \\ 0 & 4 & 0 \end{bmatrix}$ (1,3) cofactor

$= 7(0-20) - 2(0-0) + 4(0) = -140$

You can chose any row or column

Zero's, baby!

$\det A = 7 \det \begin{bmatrix} -1 & 5 \\ 4 & 0 \end{bmatrix} \cancel{- 0 \det \begin{bmatrix} 2 & 4 \\ 4 & 0 \end{bmatrix}} \cancel{+ 0 \det \begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}}$ ← cofactor expansion down 1st column

$= 7(-20) = -140$

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$$A=\begin{bmatrix} 7 & 0 & 4 & 2 & \pi \\ 0 & \pi & e & i & 10 \\ 0 & 0 & 7^2 & i & i^2 \\ 0 & 0 & 0 & 7^3 & i^3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det A = 7\det\begin{bmatrix} \pi & e & i & 10 \\ 0 & 7^2 & i & i^2 \\ 0 & 0 & 7^3 & i^3 \\ 0 & 0 & 0 & 1 \end{bmatrix} + 0$$

down first row

$$= 7\cdot\left(\pi \det\begin{bmatrix} 7^2 & i & i^2 \\ 0 & 7^3 & i^3 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= 7\pi\left(7^2\det\begin{bmatrix} 7^3 & i^3 \\ 0 & 1 \end{bmatrix}\right)$$

$$= 7(\pi)(7^2)(7^3)(1) = 7^6\pi$$

Upper Triangular

$$\begin{bmatrix} 7 & 0 & 4 & 2 & \pi \\ & \pi & e & i & 10 \\ & & 7^2 & i & i^2 \\ & \text{zeros} & & 7^3 & i^3 \\ & & & & 1 \end{bmatrix}$$

all the fun stuff is in the "upper triangle"

Main diagonal

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

signs for general (i,j) cofactor expansion

Invertable Matrices

Fun fact: consider $A \in M_{n\times n}$ upper triangular. If there's a zero on the main diagonal, then

- $\det A = 0$
- A not invertible

thus nonzero determinants detect invertibility!

Facts

- $\det A = 0$ iff A not invertable
- $\det(AB) = (\det A)(\det B)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(I_n) = 1$

$\det(A+B) \neq (\det A) + \det B$