subject: Linear Algebra

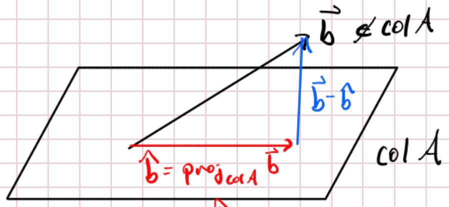
date: 15 Apr 2024

keywords: Least Squares, Square Matricies

topic: 4.7 & 5.1

Recall

$\vec{b} \notin \operatorname{col} A \iff A\vec{x} = \vec{b}$ has no solutions



$A\vec{x} = \hat{b}$ does have solution

Also, $(\vec{b} - \hat{b}) \in (\operatorname{col} A)^{\perp}$

$\Updownarrow$

$\vec{b} - \hat{b}$ is orthogonal to all columns of A

$\Updownarrow$

$(\vec{b} - \hat{b}) \cdot (\vec{a}_i) = 0$

$\Updownarrow$

$a_i^T(\vec{b} - \hat{b}) = 0$

$\Updownarrow$

$A^T(\vec{b} - \hat{b}) = 0 \iff A^T\vec{b} - A^T\hat{b} = \vec{0}$

$\Updownarrow \qquad\qquad \Updownarrow$

$\vec{b} - \hat{b} \in \ker A^T \qquad A^T\vec{b} = A^T\hat{b}$

$\Updownarrow$

$A^T\vec{b} = A^T A\hat{x}$

"normal equation"

Theorem $A\vec{x} = \vec{b}$ and $A^T A\vec{x} = A^T\vec{b}$ have the same solutions

Ex. $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ clearly, $\vec{b} \notin \operatorname{col} A$

Find least squares solutions with normal equations

$\to$ Solve $A^T A\vec{x} = A^T\vec{b}$

$$A^T A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

Fun fact: For any $A \in M_{n \times m}$, $A^T A$ and AA^T are symmetric

$$A^T\vec{b} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 5 & 3 & 3 \\ 3 & 2 & 2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & \text{smth} \\ 0 & 1 & \text{smth} \end{array}\right]$$

least squares solutions

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Def'n $S \subset X$ is invariant for $f: X \to X$ if $\forall s \in S$, $f(s) \subseteq S$

Examples $\ker A$ is invariant for $A \in M_{n \times n}$

$x \in \ker A \Rightarrow A\vec{x} = \vec{0} \in \ker A$ ✓

$A \in M$ $\vec{x}$ and $A\vec{x} = \lambda \vec{x}$ for $\lambda \in \mathbb{R}$ (i.e. scalar)

then span$\{\vec{x}\}$ invariant

Suppose $k\vec{x} \in$ span$\{\vec{x}\}$

Thus $A(k\vec{x}) = kA\vec{x} = k\lambda\vec{x} \in$ span$\{\vec{x}\}$

Def'n Let $A \in M_{n \times n}$. If $A\vec{x} = \lambda\vec{x}$ for $\vec{x} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, then $\vec{x}$ is an eigenvector with eigenvalue λ.

Moreover, span$\{\vec{x}\}$ is an eigenspace.

Ex $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has eigenvalues $\lambda = 1$, $\lambda = 2$. Find eigenvectors

Solve $A\vec{x} = \lambda\vec{x}$

$A\vec{x} = \vec{x}$ for $\lambda = 1$

$A\vec{x} - \vec{x} = \vec{0}$

$A\vec{x} - I\vec{x} = \vec{0}$

$(A - I)\vec{x} = \vec{0}$

$\vec{x} \in \ker(A - I)$

→ Eigenspace for eigenvalue λ is $\ker(A - \lambda I)$

$$A - I = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \to \begin{cases} x = z \\ y = 0 \\ z = \text{free} \end{cases}$$

$$\to \vec{x} = \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} \text{ for } z \in \mathbb{R}$$

Eigenspace for $\lambda = 1$ is span$\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$$A\begin{bmatrix} \pi \\ 0 \\ \pi \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \pi \\ 0 \\ \pi \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \\ \pi \end{bmatrix} ✓$$

Other way Given $\vec{x}$, Find λ

Easy! Compute $A\vec{x}$ and compare

$A = \begin{bmatrix} 7 & 7 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is eigenvector for some λ

$$A\vec{x} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \lambda = 7$$

Thm Eigenvectors from different eigenspaces must be independent.

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