

subject: Linear Algebra

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keywords:

topic: Change of Basis

Coordinate Vectors

$B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \subseteq \mathbb{R}^3 \qquad \vec{x} \in \mathbb{R}^3$

Find $[\vec{x}]_B$

$a\vec{b}_1 + b\vec{b}_2 + c\vec{b}_3 = \vec{x}\,; \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [\vec{x}]_B$

$\Updownarrow$

$[\vec{b}_1\ \vec{b}_2\ \vec{b}_3 \mid \vec{x}] \longleftrightarrow [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{x}$

$\downarrow$ RREF

$[\underbrace{e_1\ e_2\ e_3}_{I_3} \mid [\vec{x}]_B]$

Motivating Example

Find matrix rep w/r/t $\mathcal{E}_n$ for $T: \mathbb{R}^n \to \mathbb{R}^n$ via $T(\vec{x}) = [\vec{x}]_B$ for B a basis

$$A = [\,[e_1]_B\ [e_2]_B\ \cdots\ [e_n]_B\,]$$

A turns $\vec{x}$ into $[\vec{x}]_B$

A^{-1} turns $[\vec{x}]_B$ into $\vec{x}$

Example

V vector space, $B = \{\vec{b}_1, \dots, \vec{b}_n\} \subseteq V$; $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\} \subseteq V$

$$\begin{tikzcd} & V \arrow[dl, "{[\vec{x}]_B}"'] \arrow[dr, "{[\vec{x}]_{\mathcal{C}}}"] & \\ \mathbb{R}^n \arrow[rr, "P"'] & & \mathbb{R}^n \end{tikzcd}$$

Want $P\,[\vec{x}]_B = [\vec{x}]_{\mathcal{C}}$

Theorem: $P_{B\to\mathcal{C}} = [\,[\vec{b}_1]_{\mathcal{C}}\ [\vec{b}_2]_{\mathcal{C}}\ \cdots\ [\vec{b}_n]_{\mathcal{C}}\,]$ turns B into $\mathcal{C}$

$\Updownarrow$

$[\vec{c}_1\ \vec{c}_2\ \cdots\ \vec{c}_n \mid \vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_n]$

$\Updownarrow$ RREF

$[I \mid P_{B\to\mathcal{C}}]$

Returning to motivating Example: $P_{\mathcal{E}\to B}$: $[b_1\ \cdots\ b_n \mid \varepsilon_1\ \cdots\ \varepsilon_n]$

$\downarrow$ RREF

$[I \mid P_{\mathcal{E}\to B}]$

$[I \mid A]$

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Theorem

$$(P_{B \to C})^{-1} = P_{C \to B}$$

$$[\vec{c}_1 \cdots \vec{c}_n \mid \vec{b}_1 \cdots \vec{b}_n] \xrightarrow{RREF} [I_n \mid P_{B \to C}]$$

swap

$$[P_{B \to C} \mid I_n] \xrightarrow{RREF} [I_n \mid P_{C \to B}]$$

Ex

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} \to A_T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ (using $\mathcal{E}_2$ w/r/t dom T and codom T)

calligraphic $\to$ $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ Matrix rep w/r/t $\mathcal{E}$ in dom T and $\mathcal{B}$ in codom

standard $\to$ $B = \left[[T(e_1)]_{\mathcal{B}} \ [T(e_2)]_{\mathcal{B}} \right]$

$$\underline{P_{\mathcal{E} \to \mathcal{B}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} \ \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{B}} \right]$$

Matrix Mult.

$$\Rightarrow [\vec{b}_1 \ \vec{b}_2 \mid \vec{\mathcal{E}}_1 \ \vec{\mathcal{E}}_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 3 & 4 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -3 & -3/2 \\ 0 & 1 & 1 & -1/2 \end{array}\right]$$

$P_{\mathcal{E} \to \mathcal{B}}$