

subject: Linear Algebra
date: 5 Apr 24
topic: 4.4

keywords: identity function, inverses, Vector space, Matrix multiplication, function Composition

Schedule

No Class Mon 8
Quiz # 10 Fri 12
Test # 3 Mon 22 (?)

4.4 A Fun Theorem

$\{T : \mathbb{R}^n \to \mathbb{R}^n \text{ s.t. } T \text{ is linear}\}$ is a vector space

↳ Satisfies the 10 V.S. axioms
↳ Has 2 operations: addition and multiplication

Operations: o Function Composition

$T : \mathbb{R}^n \to \mathbb{R}^n$
$S : \mathbb{R}^n \to \mathbb{R}^n$
$(T \circ S)\vec{x} = T(S\vec{x})$

Corollary: $M_{n \times n}$ is a vector space

$\vec{a}_i, \vec{b}_i \in \mathbb{R}^n$

$T_A(\vec{x}) = A\vec{x}$ $\quad T_A : \mathbb{R}^n \circlearrowleft$ $\quad A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$

$T_B(\vec{x}) = B(\vec{x})$ $\quad T_B : \mathbb{R}^n \circlearrowleft$ $\quad B = [\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_n]$

$$(T_A \circ T_B)\vec{x} \underbrace{=}_{\text{Composition definition}} T_A(T_B(\vec{x})) \underbrace{=}_{T_B\vec{x} \text{ definition}} T_A(B\vec{x})$$

$$\xrightarrow{\text{def of } B, \vec{x}} = T_A\left([\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right)$$

$$\overset{\text{Def of M-V mult}}{=} T_A(x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots x_n\vec{b}_n)$$

$$\overset{\text{Linearity}}{=} x_1 T_A(\vec{b}_1) + x_2 T_A(\vec{b}_2) + \cdots + x_n T_A(\vec{b}_n)$$

$$\overset{\text{Def of } T_A}{=} x_1 A\vec{b}_1 + x_2 A\vec{b}_2 + \cdots + x_n A\vec{b}_n$$

$$= [A\vec{b}_1 \ A\vec{b}_2 \cdots A\vec{b}_n]\vec{x}$$

So define this Vector space addition as matrix multiplication

... barf 🤮

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$A = \begin{bmatrix} 7 & 6 & 5 \\ 2 & 7 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 \\ 7 & 0 \\ 9 & -1 \end{bmatrix}$

We require $b_i \in \mathbb{R}^3$ or $A\vec{b_i}$ not defined

$$AB = [A\vec{b_1} \;\; A\vec{b_2}] = \left[\begin{bmatrix} 7 & 6 & 5 \\ 2 & 7 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix} \;\; \begin{bmatrix} 7 & 6 & 5 \\ 2 & 7 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right] = \begin{bmatrix} 115 & -5 \\ 48 & 1 \end{bmatrix} \in M_{2\times 2}$$

What?!?

$A: \mathbb{R}^3 \to \mathbb{R}^2 \iff A \in M_{2\times 3}, \; B \in M_{3\times 2}$

$B: \mathbb{R}^2 \to \mathbb{R}^3$

$\to A\circ B = \mathbb{R}^2 \xrightarrow{B} \mathbb{R}^3 \xrightarrow{A} \mathbb{R}^2$

$A\circ B$

Have to match!

In general $A \in M_{m\times n}, \; B \in M_{n\times p}$

$\to A\circ B \in M_{m\times p}$

Suppose $A \in M_{m\times n} \quad B \in M_{n\times p}$

$$AB = [c_{ij}]$$

where $c_{ij} = \vec{a_i} \cdot \vec{b_j}$

i-th row

j-th column

inner product

as above

$$AB = \begin{bmatrix} 7 & 6 & 5 \\ 2 & 7 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 7 & 0 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 115 & -5 \\ 48 & 1 \end{bmatrix}$$

$[7\;6\;5] \cdot \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix} = 115 \qquad [2\;7\;-1] \cdot \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix} = 48$

$[7\;6\;5] \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = -5 \qquad [2\;7\;-1] \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = 1$

But also...

$BA \in M_{3\times 3} \qquad BA = \begin{bmatrix} 28 & 24 & 20 \\ 49 & 42 & 36 \\ 61 & 47 & 46 \end{bmatrix}$

$3\times 2 \quad 2\times 3$

What is the function that does nothing?

$f(x) = x$

$id(x) = x$

↳ Need $A\vec{x} = x$

Suppose $\vec{x} \in \mathbb{R}^4$, label $I_4 = \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 7 \\ 49 \\ 343 \\ 1 \end{bmatrix} \quad I_4\vec{x} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 49 \\ 343 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 49 \\ 343 \\ 1 \end{bmatrix}$ ✓