©2024 ELEANORWAISS.GITHUB.IO 

subject: Linear Algebra

date: 10 Apr 2024

keywords: Identity matrix, Invertible Matrix

topic: Invertible Matricies

Post-eclipse sadness... :(

Quiz Friday

What makes a function invertible? T is a linear function...

- One-to-one and onto (injective and surjective)
- T is an isomorphism

Quest: detect invertability

Injective: $\ker T = \{\vec{0}\}$
Pivot in every column of $A_T$
No free variables
$\forall x, y \in \text{dom } T, \quad T(x) = T(y) \Rightarrow x = y$

Surjective: $\text{imag } T = \text{codom } T$
Pivot in every row of $A_T$
$\forall y \in \text{codom } T \ \exists x \in \text{dom } T \text{ s.t. } T(x) = y$

$\rightarrow A_T$ invertable iff pivot in every row and column
$\rightarrow A_T$ must now reduce to $I_n$
$\rightarrow A_T$ must be in $M_{n \times n}$ (be a square!)

Eg $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ so $A$ invertable

New quest: Find $A^{-1}$

$A$ inv $\Rightarrow \exists A^{-1}$ s.t. $AA^{-1} = I$

Eg $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E$ Elementary matrix. Single row operation to I

$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} = EA$ whoa!

$A$ is invertible so $AA^{-1} = I_3$

$\downarrow$

$A \xrightarrow{\text{rref}} I_3$

$\hookrightarrow \exists$ row ops $E_1, \dots, E_n$ s.t. $E_1 \cdots E_n A = I$

$AA^{-1} = I_3 \Leftrightarrow \underbrace{(E_1 \cdots E_n) A}_{I_3} A^{-1} = (E_1 \cdots E_n) I_3$

$A^{-1} = E_1 \cdots E_n I_3$

$\longrightarrow \text{rref}([A|I]) = [I|A^{-1}]$ !

©2024 ELEANORWAISS.GITHUB.IO 

subject | date | keywords

topic

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}$

↳ solve $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = I$

$\rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (iff $ad-bc \neq 0$)

Ex
$$\left.\begin{aligned} x + 2y &= 5 \\ x + y &= 7 \\ z &= 27 \end{aligned}\right\}$$

$A^{-1}\begin{bmatrix} 5 \\ 7 \\ 27 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ 27 \end{bmatrix}$

↳ Solve $A\vec{x} = \vec{b}$ for $\vec{x}$

$\underbrace{A^{-1}A}_{I_3}\vec{x} = A^{-1}\vec{b}$

$I_3\,\vec{x} = A^{-1}\vec{b}$

↓ Idea

Solve for $x$: $7x = 49$

$(7^{-1})7x = (7^{-1})49$

$x = \frac{1}{7}\cdot 49 = 7$

The Invertible Matrix Theorem

(one theorem to rule them all)

One Ring to rule them all, One Ring to find them,
One Ring to bring them all and in the darkness bind them.

Let $A \in M_{n\times n}$. The following are equivalent:

1) $A$ is invertible
2) $T_A$ is an isomorphism
3) $T$ is one-to-one
4) $T$ is onto
5) $\exists A^{-1}$ s.t. $AA^{-1} = I$
6) codom $A$ = ran $A$
7) dim codom $A$ = dim col $A$
8) $\ker A = \{\vec{0}\}$
9) col $A = \mathbb{R}^n$
10) dim col $A = n$
11) imag $T = W$
12) $\ker T = \{\vec{0}\}$
13) $A$ row-reduces to $I$
14) $A$ has a pivot in every row
15) $A$ has a pivot in every column
16) $A$ has no free variables.
17) $A\vec{x} = \vec{0}$ has unique solution
18) $[A|I] \to [I|A^{-1}]$
19) $\dim(\ker T)^{\perp} = n$
20) $\dim(\ker A)^{\perp} = n$
21) $\ker A^T = \{\vec{0}\}$
22) $\forall\, b \in \mathbb{R}^n$ $Ax = b$ has a unique solution
23) row $A = \mathbb{R}^n$
24) $A^T$ is invertable

⋮