subject: Linear Algebra

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topic: 4.3

keywords: Pivot, Col A, Ker A, Solving Matrix Eqns

Solutions to Quiz #9

1) $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 4 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

2) $A\vec{x} = e_3$

$\underbrace{\begin{cases} x_1 + 3x_2 + x_3 = 0 \\ -x_2 + x_3 = 0 \\ x_1 + 4x_2 + x_3 + x_4 = 1 \end{cases}}_{\text{System of Eqns}}$ $\quad \underbrace{\left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 1 & 1 \end{array}\right]}_{\text{Augmented Matrix}}$ $\quad \underbrace{x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{Vector Eqn}}$

3) $\vec{x} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$ See above (#1)

Philosophical Discussion of "Sameness"

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix} \to \begin{array}{c} \begin{array}{cccc} P & F & P & P \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = B$$

Pivots
Free Variable

$\mathbb{R}^2 \neq \mathbb{R}^3$

Sets (not equal)

$\mathbb{R}^2 \cong \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$

$2 \neq 4$

Numbers (not Equal)

$2 \sim_2 4$

Equivalence class in $\mathbb{Z}/2\mathbb{Z}$

i.e. modular arithmetic mod 2

A is "row equivalent" to B

tilde $(\sim)$ means something else

Pivot columns of A are linearly independent columns of A

Pivot columns are a basis for col A!

$A \in M_{5\times 4}$ so $A: \mathbb{R}^4 \to \mathbb{R}^5$

$$\dim \operatorname{dom} A = \dim \operatorname{Ker} A + \dim \operatorname{col} A$$

$$4 = \underbrace{1}_{\#\text{ free columns}} + \underbrace{3}_{\#\text{ pivot columns}}$$

Finding Ker A: $A\vec{x} = \vec{0} \leftrightarrow [A|\vec{0}] \longrightarrow [B|\vec{0}]$ $\quad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$

$$\leftrightarrow \begin{cases} x - y = 0 \\ z = 0 \\ w = 0 \\ 0 = 0 \\ 0 = 0 \end{cases} \to \begin{cases} x = y \\ w = z = 0 \end{cases} \to \operatorname{Ker} A = \operatorname{Ker} B = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad$ Find $[\vec{x}]_B$

i.e. find $ab_1 + bb_2 + cb_3 = \vec{x}$ boo

instead $[\vec{b}_1\ \vec{b}_2\ \vec{b}_3 \,|\, \vec{x}] \to [I_3 \,|\, [\vec{x}]_B] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

I_3

I_n = nxn matrix, all zeros except ones along main diagonal

$I_1 = [1]$ $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ ...

Identity Matrix

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$