

subject: Linear Algebra

date: 1 Apr 2024

topic: 4.2 Understanding Systems of Matricies

keywords: Pivots, Free Variables, No. of Solutions

Planes in Space

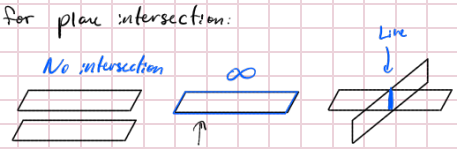
In $\mathbb{R}^3$

Subspace: $x+y+z=0$ ← Plane through the origin

Not a subspace: $5x+2y-3z=2$ ← Plane not through the origin

Options for plane intersection:

2) No intersection; ∞; Line



2 planes

3) (it's hard to draw)

	# Soln's
• Point	1
• Line	∞
• Plane	∞
• No intersection	0

Ex 1

$$\begin{matrix} x_1+x_2+x_3=5 \\ 2x_1+x_3=2 \\ x_2=10 \end{matrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -2 & -1 & -8 \\ 0 & 1 & 0 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 0 & -1 & 12 \\ 0 & 1 & 0 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -12 \end{array}\right]$$

RREF!

Need to rearrange! (not pivots yet)

What did we just get? $[I|\vec{v}] \sim x_i = v_i$

Identity matrix

i.e. $x_1=7,\ x_2=10,\ x_3=-12$

Ish, we'll get to it: $\mathbb{R}^3$

$0=1$ → Womp Womp

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivots

No Solutions (pivot in augmented column)

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{cases} x_1+2x_2=2 \\ 3x_3=4 \\ 0=0 \end{cases}$$

Pivots; No pivots; No pivots

∞ Soln's

Vacuously true

Ex 2

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{cases} x_1=3-4x_2 \\ x_3=5-x_4 \\ x_2, x_4 \text{ free} \end{cases}$$

Free Vars

Shift of a 2-d (planar) subspace within $\mathbb{R}^4$

Soln

$$\begin{cases} x_3=3/4 \\ x_1=\lambda-2x_2 \\ x_2 \text{ free} \end{cases} = \left\{\begin{bmatrix} 2-2x_2 \\ x_2 \\ 3/4 \end{bmatrix} : x_2 \in \mathbb{R}\right\}$$

Not itself a subspace, but a shifted plane in $\mathbb{R}^3$ (doesn't contain $\vec{0}$)

Kernel of a Matrix

Find $\ker A$

$$A=\begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 2 & 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & -4 & -5 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 5/4 & -1/2 & -1/4 \end{bmatrix} \sim \left[\begin{array}{ccccc} 1 & 0 & 1/2 & 1 & 3/2 & 0 \\ 0 & 1 & 5/4 & -1/2 & -1/4 & 0 \end{array}\right]$$

need row of 0s by def'n of $\ker A$

no row ops can change column of 0's

$$\ker A \sim \begin{cases} x_1=-\frac{1}{2}x_3-x_4-\frac{3}{2}x_5 \\ x_2=-\frac{5}{4}x_3+\frac{1}{2}x_4+\frac{1}{4}x_5 \\ x_3, x_4, x_5 \text{ free} \end{cases}$$

free vars

$\dim \ker A = 3$

$\ker A = \{\vec{x} : A\vec{x}=\vec{0}\}$

$$\ker A = \left\{\begin{bmatrix} -\frac{1}{2}x_3-x_4-\frac{3}{2}x_5 \\ -\frac{5}{4}x_3+\frac{1}{2}x_4+\frac{1}{4}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} : x_3, x_4, x_5 \in \mathbb{R}\right\} = \text{Span}\left\{\begin{bmatrix} -1/2 \\ -5/4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 1/4 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\} \subseteq \mathbb{R}^5$$