subject: Linear Algebra

date: 18 Mar 2024

keywords: Rowspace

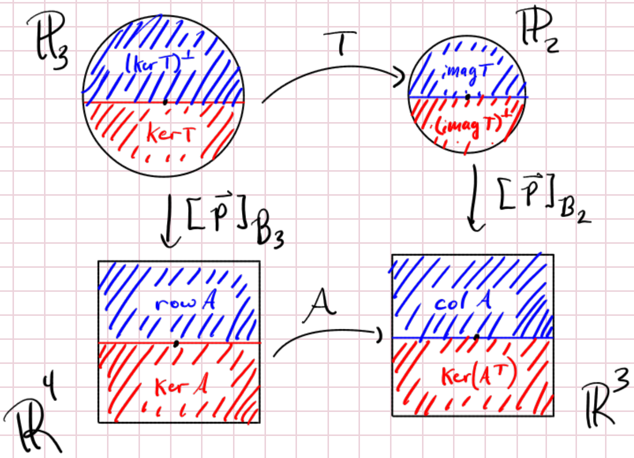
topic: 3.6 Wrapping it all together

Example

$T: \mathbb{P}^3 \to \mathbb{P}^2$ via $ax^3+bx^2+cx+d \mapsto (a-c)x^2+(b-d)x+c-a$

Using $\mathcal{B}=\{1,x,x^2,x^3\}$; $\mathcal{E}=\{1,x,x^2\}$

Build a matrix rep for A



$$A=\Big[\,[T(\vec{b}_1)]_{\mathcal{E}}\ \ [T(\vec{b}_2)]_{\mathcal{E}}\ \ [T(\vec{b}_3)]_{\mathcal{E}}\ \ [T(\vec{b}_4)]_{\mathcal{E}}\,\Big]$$

$$=\Big[\,[-x]_{\mathcal{E}}\ \ [-x^2+1]_{\mathcal{E}}\ \ [x]_{\mathcal{E}}\ \ [x^2-1]_{\mathcal{E}}\,\Big]$$

$$=\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\text{col } A = \text{span}\{\vec{a}_1,\dots,\vec{a}_4\}$$
$$=\text{span}\left\{\underset{\vec{a}_2}{\begin{bmatrix}1\\0\\-1\end{bmatrix}},\ \underset{\vec{a}_3}{\begin{bmatrix}0\\1\\0\end{bmatrix}}\right\}$$

Note: $\text{imag } T = \text{span}\{[\vec{a}_2]_B, [\vec{a}_3]_B\}$
$= \text{span}\{-x^2+1,\ x\}$

By Rank-Nullity theorem

dim ker $A = 2$ dim imag $T = 2$

dim ker $T = 2$ dim col $A = 2$

Definition For $A \in M_{m\times n}$, row $A = \text{span}\{\vec{r}_1,\dots,\vec{r}_m\} \subseteq \mathbb{R}^n$

Theorem $(\text{Row } A)^\perp \overset{?}{=} \ker A$

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Proof Sketch

Pick $\vec{x} \in \ker A \iff A\vec{x} = \vec{0} \iff x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = 0$

$$\iff \begin{bmatrix} x_1a_{11} + x_2a_{12} + \dots + x_na_{1n} \\ \vdots \\ x_ma_{m1} + \dots + x_ma_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^m$$

$\leftarrow$ Row of A

Definition

$A \in M_{m\times n}$, the transpose of A, $A^T \in M_{n\times m}$ given by $[a_{ij}] = [a^T_{ji}]$

Ex $A = \begin{bmatrix} 7 & 9 \\ 2 & 8 \\ 3 & \pi \end{bmatrix}$ $A^T = \begin{bmatrix} 7 & 2 & 3 \\ 9 & 8 & \pi \end{bmatrix}$

Theorems

If $A = M_{m\times n}$, $k \in \mathbb{R}$, $B \in M_{m\times n}$

- $(A^T)^T = A$
- $(kA)^T = k(A^T)$
- $(A+B)^T = (A^T) + (B^T)$

$(\operatorname{col} A)^\perp = \ker(A^T)$