

subject: Linear Algebra

date: 8 March 2024

topic: Matrix Representations

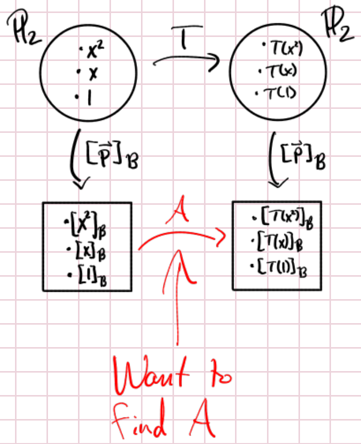
keywords: Column Space, Rank-Nullity, Kernel / Image, Conjugacy Diagrams

Recall

$T: \mathbb{P}_2 \to \mathbb{P}_2$ via $ax^2 + bx + c \mapsto (2c-b)x^2 + (a-b)$

$B = \{x^2, x, 1\}$

$T(x^2) = 1$
$T(x) = -x^2 - 1$
$T(1) = 2x^2$

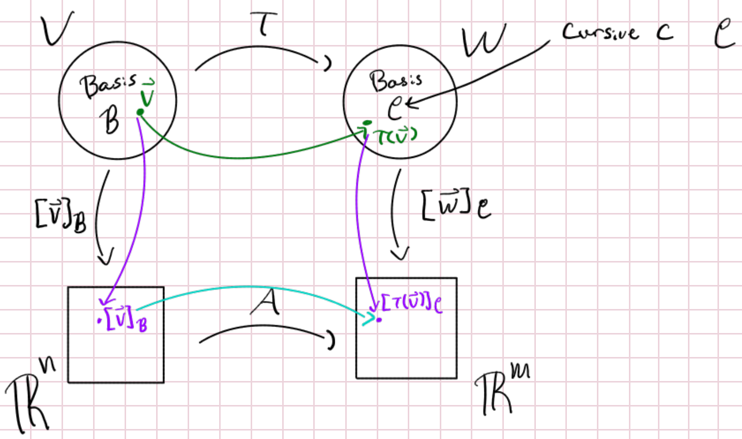


$$A = \left[[T(x^2)]_B \ [T(x)]_B \ [T(1)]_B \right]$$

$$= \left[[1]_B \ [-x^2-1]_B \ [2x^2]_B \right]$$

$$= \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \qquad A: \mathbb{R}^3 \to \mathbb{R}^3$$

... we'll come back to you



$$A = \left[[T(\vec{b}_1)]_{\mathcal{C}} \ [T(\vec{b}_2)]_{\mathcal{C}} \cdots [T(\vec{b}_m)]_{\mathcal{C}} \right]$$

Alright, back to the example above ...

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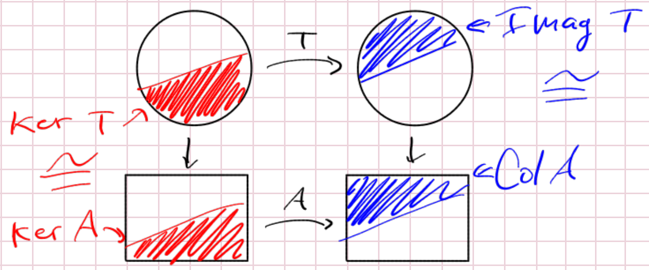
$A = \begin{bmatrix}0&-1&2\\0&0&0\\1&-1&0\end{bmatrix}$ $\quad T: \mathbb{P}_2 \to \mathbb{P}_2$ via $ax^2+bx+c \mapsto (2c-b)x^2+(a-b)$



$$\operatorname{Ker} T = \{\vec{p} \in \mathbb{P}_2 : T(\vec{p}) = \vec{0}\} = \{ax^2+bx+c : (2c-b)x^2+(a-b)=0\}$$

$$= \left\{ax^2+bx+c : \begin{matrix}2c-b=0\\ a-b=0\end{matrix}\right\} = \{ax^2+bx+c : a=b,\ 2c=b\}$$

$$= \{bx^2+bx+\tfrac{1}{2}b : b \in \mathbb{R}\} = \operatorname{span}\{x^2+x+\tfrac{1}{2}\}$$

$$\ker A = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : A\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}\right\} = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : \begin{bmatrix}0&-1&2\\0&0&0\\1&-1&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : x\begin{bmatrix}0\\0\\1\end{bmatrix} + y\begin{bmatrix}-1\\0\\-1\end{bmatrix} + z\begin{bmatrix}2\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

release her to the wild, let Y be free!!!

$$= \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : \begin{matrix}-y+2z=0\\ 0=0\\ x-y=0\end{matrix}\right\} = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : \begin{bmatrix}y\\y\\\frac{1}{2}y\end{bmatrix}, y \in \mathbb{R}\right\}$$

$$= \operatorname{span}\left\{\begin{bmatrix}1\\1\\1/2\end{bmatrix}\right\}$$

But wait... "$[\operatorname{Ker} T]_B = \operatorname{Ker} A$"

$\operatorname{Ker} T \cong \operatorname{Ker} A$

$$\operatorname{Col} A = \operatorname{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \operatorname{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}\right\}$$

Rank-Nullity: $\dim \operatorname{Ker} A = 1$

$+ \dim \operatorname{Col} A = 2$

$\dim \operatorname{dom} A = 3 \Rightarrow 3-1=2$

welcom to the higher math!

$$= \operatorname{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}\right\}$$

$$\operatorname{Imag} T = \dots = \operatorname{span}\{1, -x^2-1, 2x^2\}$$

$$= \operatorname{span}\{1, 2x^2\}$$

"$[\operatorname{Imag} T]_B = \operatorname{col} A$"