subject: Linear Algebra

date: 4 Mar 2024

keywords: Matricies

topic: 3.4) The Day the Ghosts Took Over the Board (also, Matricies)

Matricies

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [\vec{a}_1 \cdots \vec{a}_n] \in M_{m\times n}(\mathbb{R})$$

set of matricies of m rows, n columns, real numbers

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \\ 9 & \pi \end{bmatrix} \in M_{3\times 2}(\mathbb{R}) \qquad a_{21} = 3 \qquad a_{12} = 2$$

Operations on Matricies

Matrix Addition

$+: \underline{M_{m\times n}} \times \underline{M_{m\times n}} \to \underline{M_{m\times n}}$ adds componentwise

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 12 & 14 \\ 16 & 18 \end{bmatrix}$$

Scalar-Matrix

$\cdot : \mathbb{R} \times M_{m\times n} \to M_{m\times n}$ Scales everything

$$k\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} k & 2k \\ 3k & 4k \\ 5k & 6k \end{bmatrix}$$

Vector Space Stuff

$(M_{m\times n}, +, \cdot)$ is a vector space!

$$\dim M_{m\times n} = mn$$

Eg $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is standard basis for $M_{2\times 2}$

More Operations

Matrix-Vector Multiplication

$\cdot : M_{m\times n} \times \mathbb{R}^n \to \mathbb{R}^m$ via

$$A\vec{x} = [\vec{a}_1 \cdots \vec{a}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \in \mathbb{R}^m$$

$\in M_{m\times n}$ $\quad \in \mathbb{R}^n$ $\quad \vec{a}_i \in \mathbb{R}^m$ $\quad x_i \in \mathbb{R}$

i.e. the n vectors each have m components

linear combination of columns of A with weights of x

$$A = \begin{bmatrix} 7 & 7 \\ 3 & 7 \\ 0 & -7 \end{bmatrix} \in M_{3\times 2} \qquad \vec{x} = \begin{bmatrix} 4 \\ 1/7 \end{bmatrix} \in \mathbb{R}^2$$

$$A\vec{x} = \begin{bmatrix} 7 & 7 \\ 3 & 7 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} 4 \\ 1/7 \end{bmatrix} = 4\begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + \frac{1}{7}\begin{bmatrix} 7 \\ 7 \\ -7 \end{bmatrix} = \begin{bmatrix} 29 \\ 13 \\ -1 \end{bmatrix}$$

$T_A : \mathbb{R}^m \to \mathbb{R}^n$ by $T_A(\vec{x}) = A\vec{x}$ for $A \in M_{mn}$

Thus T_A is a linear transformation.
Could also write this as $A : \mathbb{R}^n \to \mathbb{R}^m$

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More Notation

Kernel $\quad \text{Ker } A = \text{Ker } T_A = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$!

Column Space $\quad \text{col } A = \text{span}\{\vec{a}_1, \dots, \vec{a}_n\} = \text{imag } A = \text{imag } T_A$!

Examples

$A = \begin{bmatrix} 7 & 4 & 7 & 5 \\ 3 & 7 & 3 & -2 \end{bmatrix} \in M_{2\times 4}$

$\text{Col } A = \text{span}\left\{\begin{bmatrix}7\\3\end{bmatrix}, \begin{bmatrix}4\\7\end{bmatrix}, \begin{bmatrix}7\\3\end{bmatrix}, \begin{bmatrix}5\\-2\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}7\\3\end{bmatrix}, \begin{bmatrix}4\\7\end{bmatrix}\right\} = \mathbb{R}^2 = \text{imag } T_A$

$\text{Ker } A = \{\vec{x} \in \mathbb{R}^4 : A\vec{x} = \vec{0}\} = \left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} : \begin{bmatrix} 7 & 4 & 7 & 5 \\ 3 & 7 & 3 & -2 \end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}\right\}$

$= \left\{\begin{bmatrix}x_1\\ \vdots \\x_4\end{bmatrix} : x_1\begin{bmatrix}7\\3\end{bmatrix} + x_2\begin{bmatrix}4\\7\end{bmatrix} + x_3\begin{bmatrix}7\\3\end{bmatrix} + x_4\begin{bmatrix}5\\-2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}\right\} = \text{smth}$

Theorem

Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ via $T_A(\vec{x}) = A\vec{x}$ for $A \in M_{m\times n}$

T_A is surjective $\iff \text{col } A = \mathbb{R}^m \iff \text{imag } A = \mathbb{R}^m$
$\iff$ cols A span $\mathbb{R}^m$

T_A is injective $\iff \text{Ker } A = \{\vec{0}\}$
$\iff$ cols A linearly independent

Ex

$A = \begin{bmatrix} 1 & 0 & -7 \\ 1 & 1 & 2 \end{bmatrix} \in M_{2\times 3}$

$\text{Ker } A = \{\vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{0} \in \mathbb{R}^2\}$

$= \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} + z\begin{bmatrix}-7\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}\right\}$

to be continued . . .