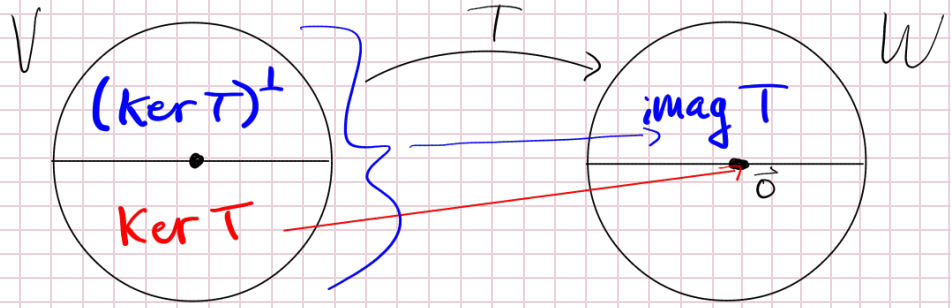
subject: Linear Algebra
date: 1 March 2024
keywords: Isomorphism, Rank-Nullity
topic: 3.3 Isomorphisms, Cont.

Pokéballs



$\begin{cases} T: V \to \text{im}\, T \quad (\text{onto}) \\ T: \ker T \to \{0\} \end{cases}$

$\Rightarrow T: (\ker T)^\perp \to \text{im}\, T$ is an isomorphism

i.e. $(\ker T)^\perp \cong \text{im}\, T$

Isomorphism Theorem the First

$\Rightarrow \dim (\ker T)^\perp = \dim (\text{im}\, T)$

Rank-Nullity Theorem

Eg. $V = \ker T \oplus (\ker T)^\perp$ (orthogonal decomposition theorem)

$\Rightarrow \dim V = \dim \ker T + \dim (\ker T)^\perp$

$\Rightarrow \dim V = \dim \ker T + \dim \text{im}\, T$

Eg. $T: \mathbb{P}_2 \to \mathbb{R}^3$ via $a_2x^2 + a_1x + a_0 \mapsto \begin{bmatrix} a_0 - a_1 \\ a_0 + a_1 \\ a_1 \end{bmatrix}$

1) Verify linearity via axioms.

2) $\ker T = \{\vec{p} : T(\vec{p}) = \vec{0}\} = \{a_2x^2 \in \mathbb{P}_2\} = \text{span}\{x^2\}$

$\Rightarrow \dim \ker T = 1 \neq 0 \Rightarrow T$ not injective

$\Rightarrow \dim V = \dim \ker T + \dim \text{im}\, T$

$\to 3 = 1 + \dim \text{im}\, T$ (This is the higher math)

$\to 3 - 1 = 2 = \dim \text{im}\, T$

$\Rightarrow \dim \text{im}\, T = 2 \neq 3 = \dim \text{codom}\, T$

$\Rightarrow T$ not surjective