

subject: Linear Algebra

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keywords: Linear Functions, Isomorphism, Coordinate Mapping

topic: 3.2/3.3 Isomorphisms

Recall

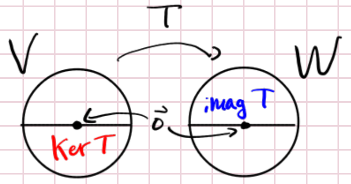
$T: \mathbb{R}^2 \to \mathbb{R}^2$ via $\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x+y \\ 0 \end{bmatrix}$

✓ linear ✗ one-to-one

✗ onto

$\ker T = \{\vec{x} : T(\vec{x}) = \vec{0}\}$

$\text{imag}\, T = \{T(\vec{x}) : \vec{x} \in \text{dom}\, T\}$



$\ker T = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$\text{imag}\, T = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

A way to build a Linear Transformation:

<u>Theorem:</u> Let $\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ be a basis for V, and $T: V \to W$ be linear. Then

Then $\text{imag}\, T = \text{span}\{T(\vec{b}_1), T(\vec{b}_2), \dots, T(\vec{b}_n)\}$

"T is entirely determined by where it maps basis vectors"

E.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, $T(\vec{u}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T(\vec{v}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Determine a formula for T.

Let $\vec{x} \in \mathbb{R}^2$, thus $\vec{x} = a\vec{u} + b\vec{v}$ (def'n of basis)

Thus $T(\vec{x}) = T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v}) = \begin{bmatrix} a-b \\ a+b \end{bmatrix}$

<u>Thm:</u> Sps T is a linear transformation. Then:

(1) T injective $\iff \ker T = \{\vec{0}\}$

(2) T surjective $\iff \text{imag}\, T = \text{codom}\, T$

Proof of (1): Suppose T injective. Since T linear, $T(\vec{0}) = \vec{0}$.

Since T injective, nothing else may map to $\vec{0}$.

Hence T injective $\Rightarrow \ker T = \{\vec{0}\}$

Suppose $\ker T = \{\vec{0}\}$. Consider $T(\vec{u}) = T(\vec{v})$.

Thus $T(\vec{u}) - T(\vec{v}) = \vec{0}$, and $T(\vec{u} - \vec{v}) = \vec{0}$.

Because $\ker T > \{\vec{0}\}$, $\vec{u} - \vec{v} = \vec{0}$. Hence $\vec{u} = \vec{v}$

Hence T injective iff $\ker T = \{\vec{0}\}$ ■

Def'n: A linear transformation that is injective and surjective is an <u>isomorphism</u>.

If $T: V \to W$, then V and W are <u>isomorphic</u>, $V \cong W$

Isomorphic vector spaces are "the same" to a certain degree subject | date | keywords

topic

E.g. $H = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ $T : H \to \mathbb{R}^2$ via $T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Note $H \subsetneq \mathbb{R}^3$ looks like a plane in 3-space.

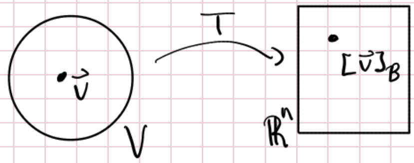
$H \neq \mathbb{R}^2$, but $H \cong \mathbb{R}^2$

3 defining qualities of an isomorphism

- Linear
- One-to-one
- Onto

E.g. let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for V, $T : V \to \mathbb{R}^n$ via

$$T(\vec{v}) = [\vec{v}]_{\mathcal{B}}$$



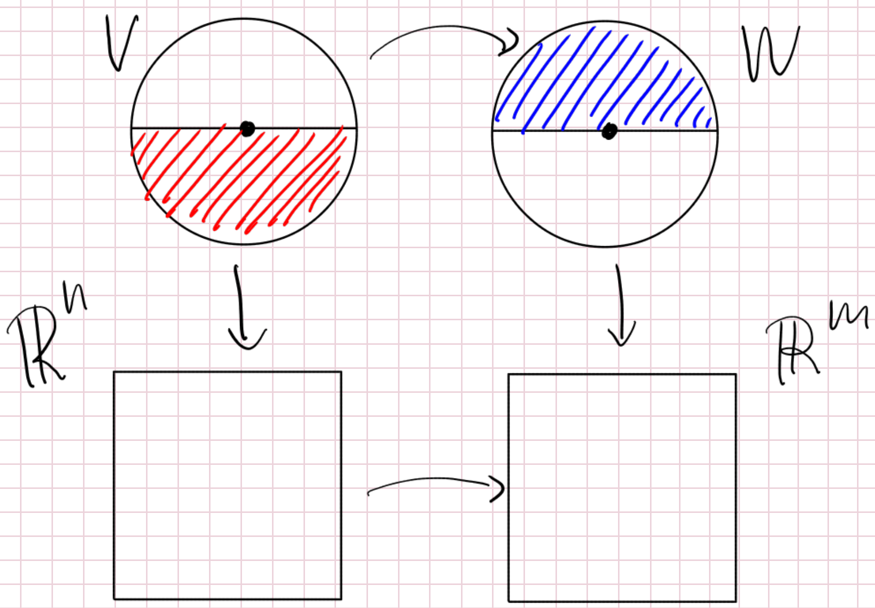
Checking for linearity is tricky, as you'll need to show $[v+u]_B = [v]_B + [u]_B$

T is known as the coordinate mapping, and is an isomorphism

Fact: need same # of basis vectors between domain and codomain for an isomorphism to work

Theorem: $V \cong W$ iff $\dim V = \dim W$ WOAH!

Ultra Pokéball



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