Teaching Lebesgue Integration to Undergraduates
Instructor's Resource Guide

A booklet to complement *The Lebesgue Integral for Undergraduates*

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Contents

Chapter 1

The Lebesgue Integral and L^1

1.1 Countable and Uncountable Infinity

Key Concepts

1. Cantor's definition of countable and uncountable sets of real numbers.
2. The Countable Union Theorem
3. Diagonalization proof that $[0, 1]$ is uncountable.

Definitions. A (nonempty) set S is:
• countable when its elements can be arranged in a sequence; that is, when S is of the form $S = \{s_1, s_2, s_3, s_4, \ldots\}$.
• uncountable when it is not countable.

Exercises

1. Use the definition to prove $\mathbb{Z} = \{\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots\}$ is countable.

$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$

$\therefore$ $\mathbb{Z}$ is countable $\blacksquare$

i.e. $s_n = \frac{n}{2}$, n even; $s_n = \frac{-n+1}{2}$, n odd $\leftarrow$ bijective $\mathbb{N} \to \mathbb{Z}$

2. Use the definition to prove the cardinality of $\mathbb{Q} \cap [0, 1]$ (the set of rationals between 0 and 1) is countably infinite.

$\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \ldots\}$ $\leftarrow$ bijection $\mathbb{N} \to \mathbb{Q}_{[0,1]}$

$\therefore$ $\mathbb{Q}_{[0,1]}$ is countable $\blacksquare$

3. Prove $[0, 1]$ is uncountable.

BWOC: Sps $[0,1]$ is countable. $\therefore [0,1] = \{s_1, s_2, \ldots\}$

Let $\sum_{a=1}^{\infty} s_{e,a} \cdot 10^{-a}$ be the decimal expansion of s_e

Thus $\sum_{a=1}^{\infty} \varphi(s_{a,a}) \cdot 10^{-a} \in [0,1]$ yet $\notin \{s_1, s_2, \ldots\}$ for "well-defined" $\varphi: \{0, \ldots, 9\}$ "endofunction"

4. Prove the **Countable Union Theorem**: A countable union of countable sets is countable. (CUCSC)

Let U_n be a countable set. Thus

$\bigcup_{n=1}^{\infty} U_n = \{U_{1,1}, U_{2,1}, U_{1,2}, U_{1,3}, U_{2,2}, U_{3,1}, U_{4,1}, U_{3,2}, \ldots\}$

and thus is countable. ∎

5. Prove the set of rationals $\mathbb{Q}$ is countable. Use the Countable Union Theorem.

$\mathbb{Q} = A_0 \cup A_1 \cup A_{-1} \cup A_2 \cup A_{-2} \cup \ldots$

where

$A_0 = \{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \ldots\}$

$A_j = \{A_{0,n} + j\}_{n=1}^{\infty}$

1.2 Measures of Intervals and Measure Zero Sets

Definitions.

- The Lebesgue measure $m(I)$ of an interval I is its length.
- A Measure Zero Set: $m(S) = 0$ when S can be covered with a sequence of open intervals $I_1, I_2, I_3, \ldots$ whose bounded total measure $\sum_{n=1}^{\infty} m(I_n)$ is arbitrarily small.

Example: $S = \{-3, 7, 31\}$ has measure zero.

Proof: Given $\varepsilon > 0$, the intervals $I_1 = (-3 - \varepsilon/6, -3 + \varepsilon/6)$, $I_2 = (7 - \varepsilon/6, 7 + \varepsilon/6)$, and $I_3 = (31 - \varepsilon/6, 31 + \varepsilon/6)$ form an open cover (do you see why $S \subseteq I_1 \cup I_2 \cup I_3$?) with bounded total measure equal to $m(I_1) + m(I_2) + m(I_3) = \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ (do you see why?).

Example: $\mathbb{N} = \{1, 2, 3, \ldots\}$ has measure zero.

Proof: Given $\varepsilon > 0$, the intervals $I_n = (n - \varepsilon/2^{n+1}, n + \varepsilon/2^{n+1})$, $n \in \mathbb{N}$, form an open cover (do you see why $\mathbb{N} \subseteq \bigcup I_n$?) with bounded total measure equal to $\sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon$ (do you see why?).

Exercises. 1. Find the length of each interval I:

1. $I = [-7, 31)$ $\quad m(I) = 31 - (-7) = 38$
2. $I = (-\pi, \sqrt{2})$ $\quad m(I) = \sqrt{2} - (-\pi) = \sqrt{2} + \pi$
3. $I = (0, \infty)$ $\quad m(I) = \infty$

2. Prove each set has measure zero.

1. $\{-6, -\sqrt{2}, 3, 31, e, \pi\}$

   Given $\varepsilon > 0$, $\bigcup_{i=1}^{6} (s_i - \frac{\varepsilon}{12}, s_i + \frac{\varepsilon}{12})$ is an open cover of S and has Lebesgue measure ε. $\blacksquare$

2. $\{2, 4, 6, 8, 10, \ldots\}$

   Given $\varepsilon > 0$, $\bigcup_{i=1}^{\infty} \left(2i - \frac{\varepsilon}{2^{i+1}}, 2i + \frac{\varepsilon}{2^{i+1}}\right)$ is an open cover of S and has Lebesgue measure ε. $\square$

3. $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$

   Given $\varepsilon > 0$, $\bigcup_{i=-\infty}^{\infty} \left(2i - \frac{\varepsilon}{2^{i+2}}, 2i - \frac{\varepsilon}{2^{i+2}}\right)$ is an open cover of S and has Lebesgue measure ε. $\square$

4. Any countable set S.

   Let $S = \{s_1, s_2, \ldots\}$. Given $\varepsilon > 0$,

   $\bigcup_{i=1}^{\infty} \left(s_i - \frac{\varepsilon}{2^{i+1}}, s_i + \frac{\varepsilon}{2^{i+1}}\right)$ is an open cover of S with Lebesgue measure ε $\square$

1.3 Step Functions and their Lebesgue Integrals

$\chi_{I_\ell} = \begin{cases} 1 & x \in I_\ell \\ 0 & x \notin I_\ell \end{cases}$

χ_{I_j}

Definitions. A step function: $\sum_{j=1}^{n} c_j \cdot \mathcal{X}_{I_j}(x)$, where c_j is a real constant and $\cancel{\mathcal{X}_{I_j}(x)}$ is a characteristic function of a bounded interval I_j.

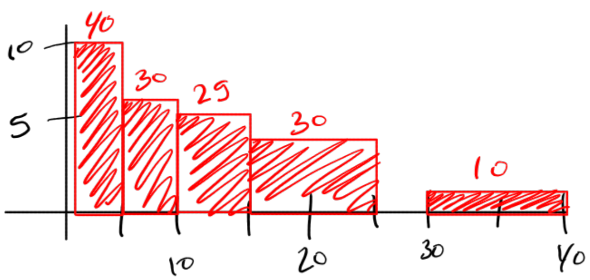
• The Lebesgue integral of a step function: $\int_{-\infty}^{\infty} \sum_{j=1}^{n} c_j \cdot \mathcal{X}_{I_j}(x)\ dx = \sum_{j=1}^{n} c_j \cdot m(I_j)$. $\leftarrow$ finite sum of bounded intervals

Exercises.

1. Why isn't $h(x) = 10 \cdot \mathcal{X}_{(1,5]} - 6 \cdot \mathcal{X}_{[4,\infty)}$ a step function? not bounded

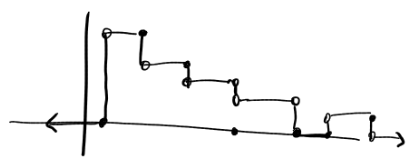
2. Graph the functions. Then find their Lebesgue integral, showing work:

   (a) $f(x) = 10 \cdot \mathcal{X}_{(1,5]} + 6 \cdot \mathcal{X}_{(5,10]} + 5 \cdot \mathcal{X}_{(10,15)} + 3 \cdot \mathcal{X}_{(15,25)} + \mathcal{X}_{(30,40]}$

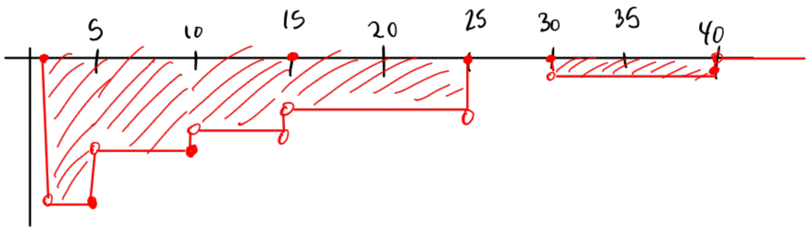


$$\int_{-\infty}^{\infty} f(x)\, d\mu = 135$$

$\longleftrightarrow$



   (b) $g(x) = -10 \cdot \mathcal{X}_{(\sqrt{2},5]} - 6 \cdot \mathcal{X}_{(5,10]} - 5 \cdot \mathcal{X}_{(10,15)} - 3 \cdot \mathcal{X}_{(15,25)} - \mathcal{X}_{(30,40]}$



3. Use the definitions to prove these theorems: (Linear Operator)

   (a) **Theorem**: If f is a step function, then so is af (where a is any real constant) and $\int a \cdot f = a \int f$.

$$f(x) = \sum_{j=1}^{n} c_j \chi_{I_j}(x) \Rightarrow a f(x) = a \sum_{j=1}^{n} c_j \chi_{I_j}(x) = \sum_{j=1}^{n} a c_j \chi_{I_j}(x)$$

$$\Rightarrow \int a f(x)\, dx = a \int f(x)\, dx$$

   (b) **Theorem**: If f and g are step functions, then so is $af + bg$ (where $a, b \in \mathbb{R}$) and $\int (a \cdot f + b \cdot g) = a \int f + b \int g$.

Note: bounds, differential can be supressed w/r/t lebesgue integral

$$\int (af + bg) = \int a \sum_{j=1}^{n} c_j \chi_{I_j} + b \sum_{j=1}^{m} c_j \chi_{I_j} = a \int f + b \int g$$

$\therefore$ lebesgue Integral is a linear operator

1.4 Limits

Key Concepts

1. *limit of a sequence* - rigorously define $\lim_{n\to\infty} s_n = L$ for a sequence of (real) numbers s_n.
2. *limit of a function* - rigorously define $\lim_{x\to a} f(x) = L$ for a (real) function f.
3. *pointwise limit of a function sequence* - rigorously define $\lim_{n\to\infty} f_n(x) = f(x)$ for a sequence of (real) functions $f_n(x)$.

1.4.1 Three Types of Limits

Definitions

1. *The limit of a sequence* $\{s_n\}$. $\lim_{n\to\infty} s_n = L$ means: given $\varepsilon > 0$, there exists $N > 0$ such that $|s_n - L| < \varepsilon$ whenever $n > N$.
2. *The limit of a function f as x approaches c.* $\lim_{x\to c} f(x) = L$ means: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$. f is continuous at c if $\lim_{x\to c} f(x) = f(c)$
3. *The pointwise limit of a function sequence*. For a specific x, $\lim_{n\to\infty} f_n(x) = f(x)$ means the sequence $\{f_n(x)\}$ approaches the value $f(x)$ in this way: given $\varepsilon > 0$, there exists $N > 0$ such that $|f_n(x) - f(x)| < \varepsilon$ whenever $n > N$. When this happens on a domain set D of x's, then the limit values $f(x)$ define a function f on D called the pointwise limit of the f_n's. Here, $\lim_{n\to\infty} f_n = f$.

1.4.2 Working with the definitions

1. Prove $\lim_{n\to\infty} \frac{2}{n+6} = 0$.

Suppose $\varepsilon>0$, let $N\in\mathbb{N}$ s.t $N> \max\{3, \frac{1}{\varepsilon}\}$
Thus $\forall\, n>N$, we have that
$\left|\frac{2}{n+6}\right| = 2\left|\frac{1}{n+6}\right| \leq 2\left|\frac{1}{2N}\right| = \frac{1}{N} < \varepsilon$ $\blacksquare$

Suppose $\varepsilon>0$, let $N\in\mathbb{N}$ s.t $N\geq \max\{3, \frac{1}{\varepsilon}\}$
Thus for all $n>N$, we have
$\left|\frac{2}{n+6}\right| = 2\left|\frac{1}{n+6}\right| < 2\left|\frac{1}{2n}\right| < \frac{1}{N} < \varepsilon$ $\blacksquare$

2. Prove $\lim_{x\to -5}(4x + 3) = -17$.

Suppose $\varepsilon>0$, Let $\delta < \frac{1}{4}\varepsilon$.
Thus for $|x-5|<\delta$ we have that
$|4x+3+17| = |4x+20| = 4|x+5| < 4\delta < \varepsilon$ $\blacksquare$

3. Apply the restriction technique to prove $\lim_{x\to 1}(x^2 + x) = 2$.

Suppose $\varepsilon>0$. Let $\delta < \min\{1, \frac{\varepsilon}{6}\}$
Thus whenever $|x-1|<\delta$, we have that

$|x-1|<\delta$
$-1<x-1<1$
$2<x+2<4 \Rightarrow |x+2|<4$

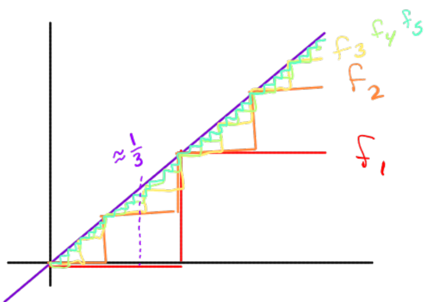
thus $|x^2+x-2| = |x+2||x-1| < 4\delta < \varepsilon$ $\blacksquare$

4. Define a sequence of functions on the interval $[0, 1]$.

Let $f_1(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}) \\ 1/2 & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$ $f_2(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{4}) \\ 1/4 & \text{if } x \in [\frac{1}{4}, \frac{1}{2}) \\ 1/2 & \text{if } x \in [\frac{1}{2}, \frac{3}{4}) \\ 3/4 & \text{if } x \in [\frac{3}{4}, 1], \end{cases}$ etc.

The nth function in the sequence is piecewise defined on 2^n intervals of equal width $1/2^n$, and it outputs the range value $k/2^n$ on the kth interval, where k runs from 0 to $2^n - 1$.

A. Graph, on the same set of axes, $f(x) = x$ along with $f_1(x)$, $f_2(x)$ and $f_3(x)$.



B. Examine $x = 1/3$. Find $\lim_{n\to\infty} f_n(\frac{1}{3})$, any way you can. Can you give an ϵ, δ proof of your result?

$\lim_{n\to\infty} f(\frac{1}{3}) = 0.010101\ldots_2 = \frac{1}{3}_{10}$

Note $\frac{1}{3} = 0.\overline{33}_{10} = 0.\overline{01}_2 = \frac{1}{2}^2 + \frac{1}{2}^4 + \frac{1}{2}^6 + \ldots = \sum_{n=1}^{\infty} (\frac{1}{2})^{2n}$

B. It turns out, for any $x \in [0, 1]$, that $|f_n(x) - x| < 1/2^n$. [Can you prove this fact?] Use it to prove $\lim_{n\to\infty} f_n(x) = x$ for any $x \in [0, 1]$. Thus $f(x) = x$ can be thought of as the pointwise limit of the sequence of functions $\{f_n(x)\}_{n=0}^{\infty}$.

Given $\varepsilon > 0$, let $N \in \mathbb{N}$ s.t $N > \frac{\ln \varepsilon}{\ln 2}$. Thus

$|f_n(x) - x| < (\frac{1}{2})^n < (\frac{1}{2})^N < \varepsilon$

1.5 Definition of L^1

Key Concepts

1. All integrals in this section refer to the LEBESGUE integral–therefore, we need to see how the Lebesgue integral is defined.
2. L^0 *functions and their integrals* - functions with finite Lebesgue integral that are the a.e. limit of a nondecreasing sequence of step functions form a collection L^0 (pronounced "L-naught").
3. L^1 *functions and their integrals* - the Lebesgue-integrable functions form a linear space L^1
4. *The standard construction* - used to form a sequence of nondecreasing step functions that a.e. approaches a given function f.
5. A property happens "almost everywhere" (a.e) when it happens everywhere except on a measure-zero set.

Definitions

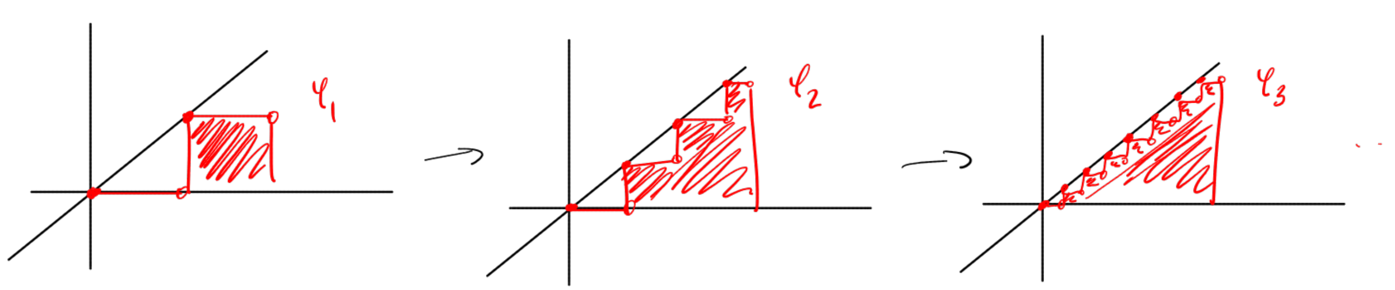
1. Suppose $\{\phi_n(x)\}$ is a nondecreasing sequence of step functions that converges pointwise almost everywhere to a function $f(x)$. Then $\int f \equiv \lim_{n\to\infty} \int \phi_n$. When this Lebesgue integral is finite, f is in the set L^0.
2. When a continuous f equals zero off an interval $[a, b]$, the standard construction automatically defines a sequence of nondecreasing step functions $\phi_n = \sum_{k=1}^{2^n} c_k \mathcal{X}_{I_k}$. It uses a "halving technique": divide $[a, b]$ into 2^n subintervals, doubling the number of subintervals for each subsequent function. The subintervals are $I_k = [a + (b-a)(k-1)/2^n, a + (b-a)k/2^n)$, and c_k is the minimum value of f on I_k (but also including the right endpoint). We always get $\lim_{n\to\infty} \phi_n = f$. We write $\lim_{n\to\infty} \int \phi_n = \int f = \int_a^b f(x)\,dx$.
3. The space L^1 of Lebesgue integrable functions consists of any function f of the form $f = g - h$, where g and h are in L^0. Its Lebesgue integral is $\int f = \int g - \int h$.
4. Throughout, these (Daniell-Riesz) definitions of the integral turn out to be well-defined; e.g., no matter what nondecreasing a.e. convergent sequence of step functions is used, the L^0 definition $\int f \equiv \lim_{n\to\infty} \int \phi_n$ always gives the same Lebesgue integral for f.

Examples

1. To find $\int_0^1 x^2\,dx$, use the standard construction to form $\phi_n(x) = \sum_{k=1}^{2^n} \left(\frac{k-1}{2^n}\right)^2 \mathcal{X}_{[\frac{k-1}{2^n}, \frac{k}{2^n})}(x)$. Reindexing, simplifying, and using a well known closed form for the sum of squares gives $\int_0^1 x^2\,dx = \lim_{n\to\infty} \int \phi_n = \lim_{n\to\infty} \frac{1}{2^{3n}} \sum_{k=0}^{2^n-1} k^2 = \lim_{n\to\infty} \frac{1}{2^{3n}} \frac{(2^n-1)2^n(2\cdot(2^n-1)+1)}{6}$. Taking the ratio of the highest-power coefficients evaluates the limit, and $\int_0^1 x^2\,dx = \frac{2}{6} = \frac{1}{3}$.
2. If $f = g - h$ with $g, h \in L^0$ and $\int g = \pi$ and $\int h = \ln 2$, what can you say about f?
   Solution: We can say $f \in L^1$ by definition, and $\int f = \pi - \ln 2$.

Exercises

1. ~~Find $\int_0^1 x^2 + x\,dx$.~~ $\int_0^2 x\,dx = \int f$ for $f = \begin{cases} x & x \in (0,2) \\ 0 & \text{else} \end{cases}$



$\int \varphi_1 = 1$ $\qquad$ $\int \varphi_2 = \frac{3}{2}$ $\qquad$ $\int \varphi_3 = \frac{7}{4}$

$\int \varphi_n = \frac{2}{2^n} \sum_{j=1}^{2^n-1} \frac{j}{2^{n-1}} = \frac{2}{2^n 2^{n-1}} \cdot \frac{(2^n-1)2^n}{2} \implies \lim_{n\to\infty} \int \varphi_n = 2$

$\int_0^1 x^2 dx$

$$\varphi_n(x) = \sum_{k=1}^{2^n} \left(\frac{k-1}{2^n}\right)^2 \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(x)$$

$$\int_0^1 x^2 dx = \lim_{n\to\infty} \int_0^1 \varphi_n(x) = \lim_{n\to\infty} \sum_{k=1}^{2^n} \left(\frac{k-1}{2^n}\right)^2 \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{2^{3n}} \cdot \frac{(2^n-1)2^n(2(2^n-1)+1)}{6} = \frac{1}{3}$$

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \frac{k^2-2k+1}{2^{n+1}} \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)} = \lim_{n\to\infty} \sum_{k=1}^{2^n} \frac{k^2}{2^{n+1}} \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)} - 2\sum_{k=1}^{2^n} \frac{k}{2^{n+1}} \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)} + \sum_{k=1}^{2^n} \frac{1}{2^{n+1}} \chi_{\left(\frac{k+1}{2^n}, \frac{k}{2^n}\right)}$$

$=$

$\int_0^1 x^2 + x\, dx \qquad \varphi_n(x) = \sum_{k=0}^{2^n} \left[\left(\frac{k-1}{2^n}\right)^2 + \left(\frac{k-1}{2^n}\right)\right] \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}$

$$\hookrightarrow = \lim_{n\to\infty} \int \varphi_n(x)dx = \lim_{n\to\infty} \frac{1}{2^n}\left(\frac{1}{2^n}\sum_{k=0}^{2^n-1} k^2 + \frac{1}{2^n}\sum_{k=0}^{2^n-1} k\right)$$

$$= \lim_{n\to\infty} \frac{1}{2^n}\left(\frac{1}{2^n}\left(\frac{(2^n-1)2^n(2(2^n-1)+1)}{6} + \frac{(2^n-1)2^n}{2}\right)\right)$$

$$= \lim_{n\to\infty} \frac{1}{2^{2n}}\left(\frac{(2^{2n}-2^n)(2^{n+1}-1)}{6\cdot 2^n} + \frac{2^{2n}-2^n}{2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2^{2n}}\left(\frac{(2^{2n}-2^n)(2^{n+1}-1) + 3(2^{2n}-2^n)2^n}{6\cdot 2^n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2^{2n+1}}\left(\frac{(2^{2n+1}-2^{n+1})(2^n+1)}{3}\right)$$

missing $\frac{1}{2^n}$ is some terms

If f inc on (a,b), then substitute left input (usually w/ reindexing)
If f dec on (a,b), then substitute right input

$$\int_0^{10} e^{-(\mu^{01}+\mu^{02})t}\mu^{02}Be^{-\delta t}dt + \int_0^{10} e^{-(\mu^{01}+\mu^{02})t}\mu^{01}\int_0^{10-t} e^{-\mu^{12}t}\mu^{12}Be^{-\delta s}ds\, e^{-\delta t}dt = B\bar{A}^{02}_{x:\overline{10}|}$$

2. Find $\int_{-1}^{1} x^2\, dx$. $= \int_{-1}^{0} x^2\,dx + \int_{0}^{1} x^2 dx = \lim_{n\to\infty} \frac{1}{2^{3n}}\left(\sum_{k=1}^{2^n} k^2 + \sum_{k=0}^{2^n-1} k^2\right)$

3. Find $\int_a^b x + 5\, dx$, where $a, b \in \mathbb{R}$. $I_k = [a + (b-a)(k-1)/2^n, a+(b-a)k/2^n]$

$\Phi_n(x) = \sum_{k=1}^{2^n} f(a+(b-a)(k-1)/2^n)\chi_{I_k}(x)$

$\lim_{n\to\infty} \int \Phi_n(x) = \lim_{n\to\infty} \overset{???}{\int} \sum_{k=1}^{2^n} (a+(b-a)(k-1)/2^n + 5)\chi_{I_k} = \lim_{n\to\infty} \frac{(b-a)}{2^n}\left[\sum_{k=1}^{2^n}(a+5) + \frac{(b-a)}{2^n}\sum_{k=1}^{2^n}(k-1)\right]$

def'n of $\int \Phi_n \chi_n$

$= \lim_{n\to\infty} \frac{(b-a)}{2^n}\left[2^n(a+5) + \frac{(b-a)}{2^n}\sum_{k=0}^{2^n-1} k\right] = \lim_{n\to\infty}\left[(a+5)(b-a) + \frac{(b-a)^2}{2^n}\left(\frac{(2^n-1)2^n}{2}\right)\right]$

4. Determine as a limit of step function integrals, but do not evaluate, $\int_0^{\pi/2} \sin x\, dx$. $= (a+5)(b-a) + \frac{(b-a)^2}{2}$

5. If $g, h \in L^0$ with $\int g = 4/5$ and $\int h = 3/4$, then what is $\int(g - h)$?

$\int (g-h) = \frac{1}{20}$, $f = g-h \in L'$

6. a) Find $\int f$, where $f = \sum_{k=0}^{11} \frac{1}{2^k}\mathcal{X}_{[k,k+1)}$

$\int \sum_{k=0}^{11} \frac{1}{2^k}\chi_{[k,k+1)} \overset{\text{def}}{=} \sum_{k=0}^{11}\frac{1}{2^k}\, m([k,k+1)) = \sum_{k=0}^{11}\frac{1}{2^k} = \frac{1-\left(\frac{1}{2}\right)^{12}}{\frac{1}{2}}$

b) Say why $f \in L^0$ and show $\int f = 2$, where $f = \sum_{k=0}^{\infty} \frac{1}{2^k}\mathcal{X}_{[k,k+1)}$

Let $\int \varphi_n(x) = \sum_{k=0}^{n} \frac{1}{2^k}\chi_{[k,k+1)} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2^k} = 2$

1.5.4(3)

Consider $\int_{-2}^{1} x^2 dx$. Compute the integral using Lebesgue integration.

define 2^n intervals on $[-2,0]$. Use right endpoints to define $\varphi(n)$

$$\int \varphi_n(x) = \sum_{k=1}^{2^n} \left(-2 + \frac{2k}{2^n}\right)^2 \cdot \frac{2}{2^n}$$

right endpoint

width of interval

left endpoint

define 2^n intervals on $[0,1]$

$$\int \phi_n = \sum_{k=1}^{2^n} \left(\frac{k-1}{2^n}\right)^2 \left(\frac{1}{2^n}\right)$$

$$= \frac{1}{(2^n)^3} \sum_{k=1}^{2^n} (k-1)^2$$

$$= \frac{1}{2^{3n}} \sum_{k=0}^{2^n-1} k^2 = \frac{1}{2^{3n}} \left(\frac{(2^n-1)(2^n)(2^{n+1}-1)}{6} \right.$$

$$\lim_{n\to\infty} \frac{2 \cdot 2^{3n} + \text{const}}{2^{3n} \cdot 6} = \frac{1}{3}$$

$$\int \varphi_n = \sum_{k=1}^{2^n} \left(\frac{2k}{2^n} - 2\right)^2 \left(\frac{2}{2^n}\right)$$

$$= \sum_{k=1}^{2^n} \left(\frac{4k^2}{2^{2n}} - \frac{8k}{2^n} + 4\right)\left(\frac{2}{2^n}\right)$$

$$= \frac{2}{2^{3n}} \sum_{k=1}^{2^n} \left(4k^2 - 8 \cdot 2^n k + 4 \cdot 2^{2n}\right)$$

$$= \frac{2}{2^{3n}} \left(4\sum_{k=1}^{2^n} k^2 - 8 \cdot 2^n \sum_{k=1}^{2^n} (k) + 2^n(4 \cdot 2^{2n})\right)$$

$$\rightsquigarrow \frac{8}{3} \Longrightarrow \int_{-2}^{1} x^2 dx = 3$$

Practice Test on Chapter 1

1. a. Prove, using the definition, that $\lim_{x \to 3} x^2 + x = 12$.

b. Is $f(x) = x^2 + x$ continuous at 3? Support your answer.

Yes! Let $\varepsilon > 0$, suppose $\delta < \min \{$

Then $\forall a$ s.t $|x-a| < \delta$, we have that $|x^2+x-a^2-a| = |x^2-a^2+x-a|$

$\leq |x-a||x+a| + |x-a|$

2. Prove, using the definition, that $\lim_{n \to \infty} \dfrac{x^2}{nx^3+1} = 0$ for all real values x.

3. a. Write **as a sequence** the elements in $S = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$ to show the set is countable.

b. Write **as a sequence** the elements in the Cartesian product set $\mathbb{N} \times \mathbb{N} = \{(m, n) : \ m, n \in \mathbb{N}\}$, to show the set is countable. Here, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

4. Recall the Cantor set is formed from a step-by-step process on the interval $[0, 1]$, where each step removes 2^n open intervals that form the middle third of any interval remaining after the nth step, where $n = 0, 1, 2, \ldots$. The points that are never removed after an infinite number of such steps form the Cantor set. **Prove the Cantor set has measure zero.** You may use the fact about Lebesgue measure on intervals: that $m(A) + m(B) = m(A \cup B)$ for any two disjoint intervals A and B.

5. Evaluate the step function integral $\int (2\pi \cdot \mathcal{X}_{[-4,4)} - 3\pi \cdot \mathcal{X}_{[-\sqrt{2},\sqrt{2})} + 4\pi \cdot \mathcal{X}_{(-1,1)})$.

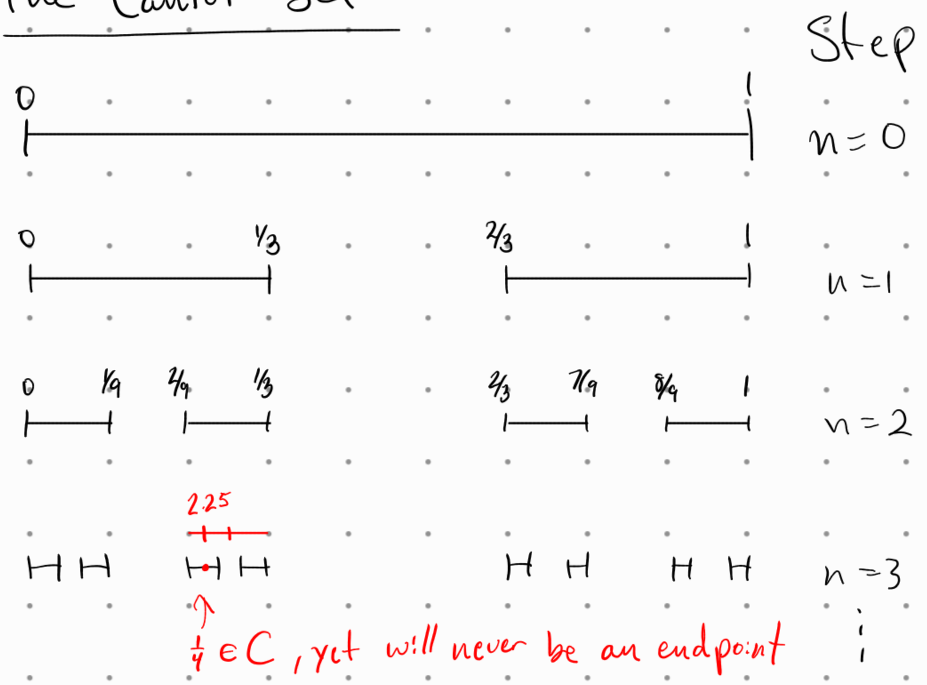
6. Using the definition of the Lebesgue integral for a function in L^0, find the Lebesgue integral $\int f$ when $f = \sum_{n=1}^{\infty} (3/4)^n \mathcal{X}_{[n,n+1)}$. Show all work. (You should not use the standard construction on this problem.)

7. Use the standard construction to evaluate the Lebesgue integral $\int_1^4 (x^2 + 1)\, dx$, explicitly giving the general formula for $\varphi_n(x)$ that forms the sequence of step functions you use. You may reference the formulas on the classroom board.

8. If $h = f - g$ is in L^1 but not in L^0, and both f and g are in L^0 with $\int f = 7$ and $\int g = -12.5$, what is $\int h$?

BONUS! Is the rational number 162/242 = .6694214876 in the Cantor set? Why or why not?

The Cantor Set



Remove the open middle third of each remaining interval

$\frac{1}{3}_{10} = 0.1_3$ $\frac{2}{3}_{10} = 0.2_3$ removed interior $x \in 0.1XXXX\ldots$ for nonzero XXXX...

$x \in C \iff x_3$ has a "1" in at least one digit

note, $\frac{1}{3}_{10} = 0.10000\ldots_3 = 0.02222\ldots_3$

Chapter 2

The Riemann Integral and Properties of the Lebesgue Integral

2.1 The Riemann Integral

Key Concepts

1. *supremum and infimum* - the least upper bound and the greatest lower bound of a real set
2. *the Lower and Upper Riemann Integral* - supremum of Lower Riemann Sums and infimum of Upper Riemann Sums of a bounded function[1] on an interval
3. *the Riemann Integral* - exists when the Lower and Upper Riemann Integrals agree, and then it equals that value (and is the definite integral you studied in calculus)
4. *Lebesgue vs. Riemann* - why the Lebesgue integral is better and how their values compare

Definitions. For a real valued set A, the (real) value $\sup A$ (called the **supremum**) must be an upper bound for A and, if M is any other upper bound for A, then $\sup A < M$. In the same way, $\inf A$ (the infimum) must be a lower bound for A and, if M is any other lower bound for A, then $\inf A > M$.

The **Lower Riemann Integral** of f on $[a,b]$, denoted $\underline{\int_a^b} f$, is the supremum of the set of all integrals of step functions bounded above by f on $[a,b]$. In other words, $\underline{\int_a^b} f = \sup \{\int \phi \ : \ \phi \text{ is a step function, and } \phi \le f\}$.

Similarly, the **Upper Riemann Integral** of f on $[a,b]$, denoted $\overline{\int_a^b} f$, is the infimum of the set of all integrals of step functions bounded below by f on $[a,b]$. In other words, $\overline{\int_a^b} f = \inf \{\int \psi \ : \ \psi \text{ is a step function, and } f \le \psi\}$.

When the Lower Riemann Integral for f equals its Upper Riemann Integral, then the **Riemann integral** $\boxed{R\text{-}\int_a^b f(x)\,dx}$ exists and equals that common value. Of course that also means that f is NOT Riemann integrable when the Lower and Upper Integrals do not agree.

Example. For a simple example, consider the Dirichlet function $\overset{\delta(x)}{\cancel{D(x)}} = \begin{cases} 1 & \text{if } x \in \overset{[0,1]\setminus\mathbb{Q}}{\cancel{\quad}} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$ for $x \in [0,1]$. $D(x)$ has a lower Riemann integral equal to 0 and an upper Riemann integral equal to 1, and hence $R\text{-}\int_0^1 D(x)\,dx = \text{DNE}$.

[1]Throughout this discussion of the Riemann integral of a function f over an interval (a,b), it will always be assumed f is bounded on the interval.

$R\underline{\int_0^1} \delta(x)dx = 0 \neq 1 = R\overline{\int_0^1} \delta(x)dx$ $\therefore$ Darboux's theorem says δ is not R-integ.

Riemann himself (in the 1860s) knew his definition failed to integrate a large number of important functions. Around 1900, Lebesgue made precise a simple condition that determined when a (bounded) function had a well-defined Riemann integral and when it did not. That description is in the following theorem.

The Riemann-Lebesgue Theorem: *A bounded function f is Riemann integrable over an interval $[a, b]$ if and only if the points of discontinuity of f on $[a, b]$ form a set of measure zero. In this case, the Riemann integral equals Lebesgue's: R-$\int_a^b f\,dx = \int_a^b f dx$.*

Darboux's Theorem: Suppose $\{\phi_n\}$ is a nondecreasing sequence of step functions and $\{\psi_n\}$ is a nonincreasing sequence of step functions, where $\phi_n \le f \le \psi_n$ for every n and where $\lim_{n\to\infty} \int_a^b \psi_n - \phi_n = 0$. Then

1. The sequences $\int_a^b \phi_n$ and $\int_a^b \psi_n$ converge to the same limit.
2. The function f is Riemann integrable.
3. We may calculate the Riemann integral of f in terms of the integrals of the step-function sequences:
$$\lim_{n\to\infty} \int_a^b \psi_n(x)\,dx = \ R\text{-}\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b \phi_n(x)\,dx.$$

We can use the Standard Construction from Chapter 1, but more generally using the infimum over a subinterval instead of the minimum, to construct the step function sequences used in Darboux's Theorem. In fact, using the supremum also constructs a nonincreasing sequence.

The Standard Construction. When attempting to form either the Riemann integral or the Lebesgue integral $\int_a^b f$ for a function $f \in L^0$, the following standard construction can be useful. Define a nondecreasing sequence of step functions $\{\phi_n\}$, where

1. Each ϕ_n is piecewise defined over $[a, b]$ using subintervals $I_k = [a + (b - a)(k - 1)/2^n, a + (b - a)k/2^n)$, $k = 1, 2, 3, \ldots, 2^n$.
2. For $x \in I_k$, the step function's value is the infimum of f over the subinterval. In short, $\phi_n(x) \equiv \inf\{f(t) : t \in I_k\}$.[2]
3. In summary, $\phi_n(x) \equiv \sum_{k=1}^{2^n} m_k \cdot \mathcal{X}_{I_k}(x)$, where $m_k = \inf\{f(t) : t \in I_k\}$ and I_k is as in part 1.
4. The "dual nonincreasing sequence" is $\psi_n(x) \equiv \sum_{k=1}^{2^n} M_k \cdot \mathcal{X}_{I_k}(x)$, where $M_k = \sup\{f(t) : t \in I_k\}$ and I_k is as in part 1.

Example. 1. Use the standard construction to determine a nondecreasing step function sequence $\{\phi_n\}$ that converges to $f(x) = 2x + 5$ on $[4, 10]$. Also find the dual sequence $\{\psi_n\}$.

Solution: Define ϕ_n and ψ_n on 2^n equal-sized subintervals of $[4, 10]$ as $\phi_n(x) = 2\cdot(4+6(k-1)/2^n)+5$ and $\psi_n(x) = 2\cdot(4+6k/2^n)+5$ for x in the kth subinterval with left endpoint $4 + 6(k - 1)/2^n$ and right endpoint $4 + 6k/2^n$, where $k = 1, 2, \ldots, 2^n$.

2. Now evaluate $\lim_{n\to\infty} \int_4^{10} \psi_n(x)\,dx$ and $\lim_{n\to\infty} \int_4^{10} \phi_n(x)\,dx$ to show $f(x) = 2x + 5$ is Riemann integrable. What is the value of R-$\int_4^{10} 2x + 5\,dx$? Justify your answer using a theorem in this section.

Solution: Using the definition of the Lebesgue integral of step functions from Chapter 1,
$\lim_{n\to\infty} \int_4^{10} \phi_n(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{2^n} [2\cdot(4+6(k-1)/2^n)+5]\cdot(6/2^n) = \lim_{n\to\infty} \frac{6}{2^n}(13\cdot 2^n + \frac{12}{2^n}\cdot\frac{(2^n-1)\cdot 2^n}{2}) = 6(13+\frac{12}{2}) = 114$. Similarly,
$\lim_{n\to\infty} \int_4^{10} \psi_n(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{2^n} [2\cdot(4+6k/2^n)+5]\cdot(6/2^n) = \lim_{n\to\infty} \frac{6}{2^n}(13\cdot 2^n + \frac{12}{2^n}\cdot\frac{(2^n\cdot(2^n+1)}{2}) = 6(13+\frac{12}{2}) = 114$.

Since the limits agree, Darboux's Theorem guarantees the Riemann integral exists and is the common limit: R-$\int_4^{10} 2x + 5\,dx = 114$.

[2] Remember: as we study Riemann integrals, we assume f is bounded on $[a, b]$, and so the infima will exist.

Exercises.

1. Find the step-function sequences $\{\phi_n\}$ and $\{\psi_n\}$ from the standard construction and its dual sequence for the function $f(x) = 3x+1$ on the interval $[0,2]$. Note f is continuous and therefore (as the Riemann-Lebesgue Theorem implies) Riemann integrable. Evaluate $\int \phi_n$ and $\int \psi_n$ as closed-form expressions in n, and show $\lim_{n\to\infty} \int \phi_n = \lim_{n\to\infty} \int \psi_n$. Then use Darboux's theorem to find the value of R-$\int_0^2 3x+1\,dx$.

2. Repeat the last exercise, but use $f(x) = x^2$ on the interval $[1,2]$.

3. Determine points of discontinuity for $f(x) = (x-1)(x^2+5x+4)/(x^2-1)$. What does the Riemann-Lebesgue theorem say about f? Use that theorem to find R-$\int_{-10}^{10} f(x)\,dx$.

2.2 Properties of the Lebesgue Integral

Key Concepts

1. *monotone sequence of functions*

2. *limits of integrals* When can you pass a limit through an integral and get an equal value? When using the Lebesgue integral, the Monotone Convergence Theorem provides one setting. But when using the Riemann integral, equality is not guaranteed.

3. *the truncation method* create a monotone sequence approaching any nonnegative function f by truncating f at $x = n$.

The Monotone Convergence Theorem (Beppo Levi): Assume $\{f_n\}$ is a monotone sequence of functions in L^1, where the integrals collectively satisfy $|\int f_n| \leq M$ for some finite upper bound M. Then the sequence $\{f_n\}$ converges a.e. to a function f in L^1, and

$$\int \lim_{n\to\infty} f_n = \int f = \lim_{n\to\infty} \int f_n.$$

Examples. 1. Use the monotone convergence theorem to find $\int_0^\infty \mathrm{e}^{-x}\, dx$.
Solution: Truncating a nonnegative function at $x = n$ to form a sequence function f_n always produces a nondecreasing (monotone) sequence, because $f_n(x) = f_{n+1}(x)$ for $x \leq n$ or $x > n+1$ (where both functions equal 0), and $f_n(x) = 0 \leq f_{n+1}(x)$ for $x \in (n, n+1]$. This **truncation method** applies here: set $f_n(x) = \mathrm{e}^{-x}$ for $x \in [0, n]$ and $f_n(x) = 0$ for $x > n$. (Equivalently, $f_n(x) = \mathcal{X}_{[0,n]}(x) \cdot \mathrm{e}^{-x}$.) By the Monotone Convergence Theorem, $\int_0^\infty \mathrm{e}^{-x}\, dx = \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_0^n \mathrm{e}^{-x}\, dx = \lim_{n\to\infty} -\mathrm{e}^{-x}\Big|_0^n = \lim_{n\to\infty} 1 - \mathrm{e}^{-n} = 1$.

2. Enumerate the rationals on $[0, 1]$, writing them as $a_1, a_2, \ldots$. Define $f_n(x) = \begin{cases} 1 & x = a_1, a_2, \ldots, a_n \\ 0 & \text{elsewise} \end{cases}$ What does the Monotone Convergence Theorem say about the integrals of this function sequence? Does that result hold if the Riemann integral is used instead of Lebesgue?
Solution: The sequence $f_n(x)$ is monotone on $[0, 1]$ because $f_n(x) = f_{n+1}(x)$ for all x except at $x = a_{n+1}$, where $f_n(a_{n+1}) = 0 < 1 = f_{n+1}(a_{n+1})$. By the Monotone Convergence Theorem, $\int f \equiv \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int 0 = 0$, because each f_n equals zero a.e. (except at a finite number of rational values). Of course, the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ is the Dirichlet function, because it equals 1 at each rational and 0 at each irrational. The result does not hold if we were to use the Riemann integral instead. In fact, we have seen that the Dirichlet function is not Riemann integrable, even though the Riemann integral of each f_n is (each f_n is continuous except at a finite number of rational values, hence is Riemann integrable by the Riemann-Lebesgue Theorem). It is then easy to realize that R-$\int f_n = 0$, but R-$\int f \neq 0 = \lim_{n\to\infty}$ R-$\int f_n$.

Other Important Theorems. 1. Riemann integrable functions f often have Riemann (definite) integrals found using the Fundamental Theorem of Calculus, by finding the antiderivative and evaluating at the integrand endpoints. Any such function is also automatically Lebesgue integrable, and so the Fundamental Theorem applies the same way to find the Lebesgue integral of f. Actually, there is a slightly more general version of the Fundamental Theorem for the Lebesgue integral: 

The L^1 Fundamental Theorem of Calculus: For f in $L^1[a, b]$, define $F(x) = \int_a^x f(t)\, dt$ for $x \in [a, b]$, and $F(x) = 0$ elsewhere.

A. Then $F'(x) = f(x)$ at a point $x \in [a, b]$ where f is continuous.[3]

B. If f is continuous on $[a, b]$ and F is now any antiderivative of f (i.e., $F'(x) = f(x)$ on $[a, b]$), then $\int_a^b f(x)\, dx = F(b) - F(a)$.

For example, $\int_0^\pi \sin x\, dx$ is easily evaluated as $\int_0^\pi \sin x\, dx = -\cos x\Big|_0^\pi = \cos 0 - \cos \pi = 2$. Of course, when finding antiderivatives, just as in Calculus, techniques such as u-substitution are helpful and important.

[3] The derivative and continuity at the interval's endpoints a and b are understood to be one-sided concepts. For example, $F'(a)$ refers to the right-hand derivative, where the limit of the difference quotient $(F(a + h) - F(a))/h$ is taken by considering only values of h that are positive and tending toward 0. Similarly, continuity of f at b means, given $\varepsilon > 0$, there exists δ such that $|f(x) - f(b)| < \varepsilon$ whenever $0 \leq b - x < \delta$ (in other words, we consider only x values to the left of b).

2. **Integration by Parts:** Assume the functions u and v are in $L^1[a,b]$ and the functions U and V are defined, for any $x \in [a,b]$, as $U(x) = \int_a^x u(t)\,dt + C_1$ and $V(x) = \int_a^x v(t)\,dt + C_2$, where C_1 and C_2 are arbitrary constants. Then $\int_a^b U\,v = U \cdot V\big|_a^b - \int_a^b V\,u$.

For example, $\int_1^2 x \ln x\,dx$ is easily evaluated using $U(x) = \ln x$ and $v(x) = x$.

$u = \frac{1}{x}$ $\quad V = \int_0^x t\,dt = \frac{1}{2}x^2$

Exercises.

1. Use the Fundamental Theorem and/or Integration by Parts to evaluate the Lebesgue integrals:

   (a) $\int_0^1 (2x+5)^5\,dx$

   $u = 2x+5$
   $du = 2dx \to dx = \frac{1}{2}du$

   $$\frac{1}{2}\int_{x=0}^{x=1} u^5\,du = \frac{1}{12}(2x+5)^6\Big|_0^1 = \frac{7^6}{12}$$

   (b) $\int_0^\pi x \sin x\,dx$ $\quad = \left[-x\cos x + \sin x\right]_0^\pi = (-\pi\cos\pi + \sin\pi) - (\cancel{\sin 0}) = \pi$

   π

$+\ x$	$\sin x$
$-\ 1$	$-\cos x$
$+\ 0$	$-\sin x$

2. Apply the Monotone Convergence Theorem to evaluate $\int_2^\infty x^{-2}\,dx$. $\overset{\text{FTC Riemann}}{=} -x^{-1}\Big|_2^\infty = \frac{1}{2}$

Consider $\{\varphi_n : [2, 2+n] \to \mathbb{R}$

where $\int \varphi_n = \int_2^{2+n} x^{-2}\,dx$ $\quad x^{-2} \geq 0\ \forall x$

Note $\int \varphi_n = \int_2^{2+n} x^{-2}\,dx \leq \int_2^{2+n+1} x^{-2}\,dx = \int \varphi_{n+1}$, thus

$\int \varphi_n$ is monotonically increasing.

Also note that $\lim_{n\to\infty} \int \varphi_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2}$

$$\therefore \int \lim_{n\to\infty} \varphi_n \overset{\text{MCT}}{=} \lim_{n\to\infty} \int \varphi_n = \frac{1}{2}$$

2.3 The Lebesgue Dominated Convergence Theorem

Key Concepts

1. *dominated sequence of functions* A sequence $\{f_n\}$ is dominated by a function $g \in L^1$ when $|f_n| \leq g$ for all n.

2. *limits of integrals* When can you pass a limit through an integral and get an equal value? When using the Lebesgue integral, the Lebesgue Dominated Convergence Theorem (LDCT) provides a setting in addition to the Monotone Convergence Theorem (discussed in the last section). But when using the Riemann integral, equality is not guaranteed.

The Lebesgue Dominated Convergence Theorem: Suppose $\{f_n\}$ is a sequence of functions in L^1 that converges almost everywhere to a function f. Also suppose the sequence functions are dominated by an integrable function g, in the sense that $|f_n| \leq g$ for all n, where $g \in L^1$. Then $f \in L^1$ and its Lebesgue integral can be evaluated by passing the limit through the integral:

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = \int f.$$

Example.

Write $\lim\limits_{n\to\infty} \int_1^n \cos(x^{-1})/x^2 \, dx$ as a Lebesgue integral of an integrable function, without a limit sign. Find the integral's value.

Solution: Define $f_n(x) = \mathcal{X}_{[1,n)}(x)\cos(x^{-1})/x^2$, which is integrable because it is continuous a.e. and dominated by the integrable function $g(x) = x^{-2} \cdot \mathcal{X}_{[\cancel{0}1,\infty)}(x)$. By the LDCT, $\lim\limits_{n\to\infty} \int_1^n \cos(x^{-1})/x^2 \, dx = \int_1^\infty \cos(x^{-1})/x^2 \, dx$. Substituting $u = x^{-1}$, $\lim\limits_{n\to\infty} \int_{1/n}^1 \cos u \, du = \lim\limits_{n\to\infty} \sin u\Big|_{1/n}^1 = \sin 1 - \sin 0 = \sin 1$.

Exercises. 1. Use either the Monotone Convergence Theorem or the Lebesgue Dominated Convergence Theorem (LDCT) to evaluate $\lim\limits_{n\to\infty} \int_0^1 (1+x/n)^{-2n} \cos(x/n) \, dx$. Make sure you justify your use of your choice of Theorem.
(Hint: $\lim\limits_{\cancel{x}\,n\to\infty} (1+x/n)^{-2n} = e^{-2x}$.)

$\int_0^1 e^{-2x} \cos(0)\, dx = \int_0^1 e^{-2x}\, dx = \frac{e^{-2x}}{-2}\Big|_0^1 = \frac{1}{2}(1-e^{-2})$

2. Use the LDCT to evaluate $\int_0^\infty e^{-x} \cos x \, dx$. Make sure you properly justify use of the theorem.

Linear Algebra: study of linear transformations on finite dimensional vector space

Vector Space: A set $\{f_i\}_{i=1}^{\infty}$ endowed with addition and scalar multiplication such that certain axioms hold

Ex: $\mathbb{P}_1$ has dimension 2

$\dim V = |B|$ given B is a basis for V

This basis endows a norm: $\mathcal{E}_{\mathbb{P}_1} = \{e_1, e_2\} = \{1, x\}$

$\|f\| \geq 0$, with equality iff $f = \vec{0}$

$\|f+g\| \leq \|f\| + \|g\|$

e.g. $\|a+bx\| = \sqrt{a^2+b^2}$ ← Euclidean Norm (ℓ^2???)

$\mathcal{E}_{\mathbb{P}_1}$ is an orthonormal basis w/r/t $\mathbb{R}^2$

i.e. $\|e_1\| = 1$, $\|e_2\| = 1$, $[e_1]_{\mathbb{R}^2} \cdot [e_2]_{\mathbb{R}^2} = 0$

Isomorphisms: $\mathbb{P}_1 \cong \mathbb{R}^2$

Examples of linear functions on functions

- Derivative
- Integrals
- Expected Values
- Composition

2.4 Fourier Series

$\|f\| \overset{def}{=} \left(\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx\right)^{1/2} \leftrightarrow \|f\|_\mu^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,d\mu$

Key Concepts

Recall $\sin u \sin v = \frac{1}{2}(\cos(u-v) - \cos(u+v))$ $\rightarrow \sin^2 u = \frac{1}{2}(1-\cos 2x)$

e.g. $|\sin x|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos 2x)\,dx = \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2x\right]_{-\pi}^{\pi} = 1$ ✓

1. *Trigonometric Series* A finite or infinite sum of terms that are constant multiples of $\sin nx$ or $\cos nx$, $n = 0, 1, 2, \ldots$. Such series naturally represent a function periodic with period 2π, and so they are naturally studied over the interval $[-\pi, \pi]$ and repeated periodically over other intervals of length 2π..

basis vectors: $\left\{\frac{1}{\sqrt{2}}, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots\right\}$

2. *Series Coefficients* In a series, the constant multipliers of the trigonometric functions.

3. *Bounded Variation* f is of bounded variation when it can be expressed as the difference $g - h$ of two nondecreasing bounded functions.

inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\overline{g(y)}\,dy$ ← Complex conjugate

4. *Convergence of Fourier Series* When does a Fourier series of a function converge, and to what value does it converge?

$\langle \overbrace{\sin x, \cos x}^{real}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(x)\cos(x)\,dx = 0$

Definitions.

1. The (classical) **Fourier series** is of the form $a_0 + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$ and is often called a **trigonometric series** because its components are sine and cosine functions. Writing a function $f(x)$ as a Fourier series gives insights into the function's behavior and is important to describe many physical properties in the world around us.

2. There are formulas for a function's **Fourier coefficients** a_0 (the constant coefficient), a_n and b_n, for $n = 1, 2, 3, \ldots$. These are, for a function f: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$.

3. The (nth) **Dirichlet kernel** is the function $D_n(x) = 1/2 + \sum_{k=1}^{n}\cos kx$, $x \in \mathbb{R}$. The **Dirichlet-Jordan Theorem** then says: When $f \in L^1(-\pi, \pi)$ is periodic with period 2π and, for $x \in \mathbb{R}$, has **bounded variation** on an interval $[x-h, x+h]$, where $0 < h \leq \pi$, then the Fourier series of f converges at x to
$\lim_{t\to 0^+}(f(x+t) + f(x-t))/2$.

Example. Find the Fourier series for $f(x) = x$, $-\pi < x \leq \pi$ (and where $f(x)$ is repeated periodically over other intervals of length 2π).

Solution: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,dt = \frac{1}{2\pi}\frac{t^2}{2}\Big|_{-\pi}^{\pi} = 0$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt = \frac{1}{n\pi}(t\sin nt + \frac{1}{n}\cos nt)\Big|_{-\pi}^{\pi} = 0$, and

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{1}{n\pi}(-t\cos nt + \frac{1}{n}\sin nt)\Big|_{-\pi}^{\pi} = \frac{2(-1)^{n+1}}{n}$ for $n = 1, 2, 3, \ldots$.

From the Fourier series formula, the series is then $2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \ldots = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$.

↙ fourier series of sum is sum of fourier series

Exercises.

1. Find the Fourier series for $f(x) = x^2 + 1$, $x \in (-\pi, \pi]$.

for $\varphi(x) = x^2$, $a_k = \langle \varphi(x), \cos(kx)\rangle = \frac{1}{2\pi}\left[\frac{x^2}{k}\cancel{\sin(kx)} + \frac{2x}{k^2}\cos(kx) - \frac{2}{k^3}\cancel{\sin(kx)}\right]_{-\pi}^{\pi} = \frac{2(-1)^k}{k^2}$

$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\cos kx\,dx$

$$\begin{cases} +x^2 & \cos kx \\ -2x & \frac{1}{k}\sin kx \\ +2 & -\frac{1}{k^2}\cos kx \\ -0 & -\frac{1}{k^3}\sin kx \end{cases}$$

$\Rightarrow x^2 + 1 = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cdot 2\cos(nx)$

2. Show that $f(x) = x^2 + 1$, $x \in (-\pi, \pi]$ satisfies the hypotheses of the Dirichlet-Jordan Theorem. In other words, show $f(x) = x^2 + 1 \in L^1(-\pi, \pi)$ and is of bounded variation.

3. What does the Fourier series (from Exercise 1, above) for $f(x) = x^2 + 1$, $x \in (-\pi, \pi]$ converge to at $x = 0$? What about at $x = \pi$?

4. From the Example and the answer to Exercise 1, above, what can you say is the Fourier series for $f(x) = x^2 + x + 1$, $x \in (-\pi, \pi]$?

$$f(x) = (x^2+1) + x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) + \left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \right)$$

Practice Test on Chapter 2

1. Prove **Theorem 2.2.4:** If $f \in L^1$, then so is $|f|$, and $|\int f| \leq \int |f|$.

2. Use the Monotone Convergence Theorem, u-substitution, and the Fundamental Theorem of Calculus to evaluate $\int_{-\infty}^{0} \dfrac{x}{(1+x^2)^2}\, dx$. Make sure you justify the use of the Monotone Convergence Theorem.

3. This problem studies the Riemann integral of $f(x) = x^2 + 1$.

a. Give a simple property of this function that guarantees its Riemann integral from 0 to 2 exists and is finite.

b. Use the standard construction to evaluate the Riemann integral R-$\int_0^2 x^2 + 1\, dx$, explicitly giving $\varphi_n(x)$ and $\psi_n(x)$ that form the sequences of step functions you use, and then evaluating $\lim_{n\to\infty} \int \varphi_n(x)\, dx$. (You do NOT need to evaluate the limit of the integrals of the dual sequence ψ_n.)

4. Use the Lebesgue Dominated Convergence Theorem and integration by parts twice to evaluate $\int_0^\infty 2\mathrm{e}^{-x} \cos x\, dx$. Make sure you justify the use of the theorem.

5.Evaluate, justifying each step, the following limit: $\lim\limits_{n\to\infty} \int_0^\infty (1+nx^2)(1+x^2)^{-n}\, dx$.

6. Find the Fourier series for $f(x) = |5x|$ (as it is defined on the interval $(-\pi, \pi]$ and extended periodically with period 2π off of this interval).

subject: Lebesgue Theory

date: 26 Feb 2024

topic: Fourier Series and L^2

keywords: Fourier Series, Inner Product, Even/odd Functions, Norm, Function Spaces, L^1, L^2

Example

Consider $f(x) = x$ (note x is odd, thus $a_n \equiv 0$)

$$b_n = \langle x, \sin nx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx = \left(\frac{-x}{n}\cos nx + \cancel{\frac{1}{n^2}\sin nx} \Big|_{-\pi}^{\pi} \right) =$$

$+\ x$	$\sin nx$
$-\ 1$	$\frac{-1}{n}\cos nx$
$+\ 0$	$\frac{-1}{n^2}\sin nx$

$$= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos n\pi - \frac{\pi}{n}\cos n\pi\right)$$

$$= \frac{-2}{n}\cos n\pi = \frac{2}{n}(-1)^{n+1}$$

$$\Rightarrow \boxed{x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)}$$

$$x = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2/3 \\ -1/2 \\ \vdots \end{bmatrix}$$

Properties of L^1

L^1 is endowed with a norm: $\varphi \in L^1 \Rightarrow \|\varphi\|_{L^1} = \int |\varphi|\,d\mu$

... but no inner product.

$\|\varphi\|_{L^1}$ meets all the properties of a norm

- $\|\varphi\|_{L^1} \geq 0 \quad \forall \varphi \in L^1$
- $\|\varphi\| = 0 \iff \varphi \equiv 0$
- $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|$
- L^1 is complete (contains limiting points)

Chapter 3

Function Spaces

3.1 The spaces L^p

Key Concepts

1. *metric space* - a set with a concept of distance (norm) satisfying certain properties
2. *Banach (Hilbert) spaces* - normed vector spaces that are complete (with an inner product), specifically the L^p *spaces*

   ↳ limits of Cauchy sequences converge w/i the space
3. *nonmeasurable sets* - and properties of measurability
4. *measurable functions* and their connection to measurable sets
5. *the mid function* and its application to proofs on L^p spaces
6. *Hölder's Inequality* and the *Riesz-Fischer Theorem*

L^2 is the only Hilbert space (i.e. w/ norm)

L^p are Banach spaces

3.1.1 Measurable Functions

A Nonmeasurable Set—the Vitali Set (See [3, p. 127]) The Lebesgue measure of a set requires the following three properties:

1. The integral of a characteristic function $\int \mathcal{X}_A$ must equal $m(A)$ for any set A
2. *Countably Additive:* For any two disjoint sets, $m(A) + m(B) = m(A \cup B)$; in fact, we require such a property holds for any countable number of disjoint sets.
3. *Translation Invariance:* For any set A, $m(A) = m(A + t)$ for a set A and real number t, where $A + t$ is the translation set of A, defined as $A + t = \{a + t : a \in A\}$.

If a set breaks any one of these, the set is *nonmeasurable*. Its characteristic function $\mathcal{X}_A$ will then not behave well in terms of the integral; it is a *nonmeasurable function.* An example is the Vitali set V:

1. Partition the interval $[0, 1]$ into an uncountable number of subsets
2. Two numbers v and w in $[0, 1]$ get into the same subset when $v - w$ is a rational number. (e.g., $Q \cap [0, 1]$ form one subset.)
3. Every value in $[0, 1]$ is in exactly one of the subsets.
4. Choose exactly one element from each of these subsets, and collect the choices into a set V.
5. This set is not measurable!

To see why there is a problem:

1. List the rationals in $[0, 1]$ as $s_1, s_2, s_3, \dots$.

2. Take any one of these numbers s_j and form the set $S_j = \{s \oplus s_j : s \in V\}$, where

$$s \oplus s_j = \begin{cases} s + s_j & \text{if } s + s_j \leq 1 \\ s + s_j - 1 & \text{if } s + s_j > 1. \end{cases}$$

3. The modular sum makes $0 \leq s + s_j \leq 1$, and so each S_j is a subset of $[0, 1]$.

4. Each $x \in [0, 1]$ is in exactly one S_j. (Can you prove this?)

Putting the previous points together implies

$$[0, 1] = \bigcup_{j=1}^{\infty} S_j$$

for disjoint sets S_j. Each set S_j is a translation set of V, and so by translation invariance we should have $m(S_j) = m(V)$, $j = 1, 2, 3, \dots$. Then by countable additivity, we should have

$$1 = m([0, 1]) = m(\bigcup_{j=1}^{\infty} S_j) = \sum_{j=1}^{\infty} m(S_j) = \sum_{j=1}^{\infty} m(V).$$

No matter what measure we might assign to $m(V)$, we have problems. If $m(V) = 0$, then this equation would imply $1 = 0$. If $m(V)$ is a positive or negative value, then the summation does not converge to a finite number, and so there would be no way for it to equal 1. It's impossible for the set V to have a correctly assigned measure. V is a nonmeasurable set. And then the characteristic function $f(x) = \mathcal{X}_V(x)$ is nonmeasurable, because $\int \mathcal{X}_V(x)\, dx = m(V)$ does not exist. Since $m(V)$ is badly behaved, so is $\mathcal{X}_V$. ■

Our goal this chapter to is to consider spaces of functions, specifically *measurable functions*:

Definition: A function is *measurable* if and only if for every measurable set S, the preimage $f^{-1}(S)$ is a measurable set.

Now we see the problem: the Vitali set V was the preimage of $T = \{1\}$ under the function $\mathcal{X}_V$. T is measurable (it has measure zero), so $f^{-1}(T)$ should be measurable, but it isn't. A function f that has this problem will not integrate correctly. These badly behaved functions are the *nonmeasurable functions*.

Which functions are measurable?

Remember that we previously defined the space L^0 as

$$L^0 = \{f : \text{ there exists a sequence of step functions } \phi_n \text{ such that } \phi_n \to f \text{ a.e. and } \int f = \lim \int \phi_n < \infty\},$$

and then L^1 as the set of Lebesgue integral functions, i.e. $L^1 = \{f : f = g - h \text{ with } g, h \in L^0\}$.

Now we want to characterize this set in terms of measurable functions. From the previous section, we see that not even all of the characteristic functions are measurable. How do we determine which functions *are* measurable, other than the definition?

Definition: For real valued functions f and g with $g > 0$, the function mid$\{-g, f, g\}$ is defined as having the unique range value chosen from the three output values $-g, f, g$ (not necessarily distinct) that is between the other two.

Definition: A real function f is measurable if and only if mid$\{-g, f, g\} \in L^1$ for every nonnegative function $g \in L^1$.

Exercise. Use the mid function to prove the following properties.

Theorem: For any real-valued function f:

1. If f is integrable, then f is measurable.
2. If f is continuous, then f is measurable.
3. If f is measurable, then so is $|f|$.
4. If f is measurable, then so is $|f|^p \; \forall p \in \mathbb{N}$. but also $p \in \mathbb{Z}_{\geq 0}$
5. If f is measurable and $|f| \leq g$ for g integrable, then f is integrable.
6. If $|f|$ is integrable, then so is f.

3.1.2 Definition of L^p

The space L^1 has several properties:

1. It is a *vector space* - if you add two functions in L^1, or multiply by a scalar, you stay in L^1.
2. It is a *metric space* - the **norm** given by $\|f\| = \int |f|$ gives a sense of *size* (similar to absolute value) for functions. Similarly, $\|f - g\|$ is the *distance* between two functions.
3. It is a *Banach space*, because in addition to the first two properties, it is *complete*: for any sequence of functions f_n belonging to L^1, if $\|f_n - f\| \to 0$, then $f \in L^1$. This means you can't "leave" the space through limits of sequences of functions.

Definition: The space L^p for real $p \geq 1$ is defined as

$L^1 = \{f : f \text{ is measurable}, \|f\|_1 = \int |f| \, d\mu < \infty\}$

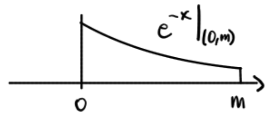
$L^2 = \{f : f \text{ is measureable}, \|f\|_2 = \left(\int |f|^2 d\mu\right)^{1/2} < \infty\}$

$$L^p = \{f : f \text{ is measurable and } \|f\|_p = \left(\int |f|^p\right)^{1/p} < \infty\}.$$

L^p spaces make no distinction between two functions that are equal almost everywhere. All L^p spaces are Banach spaces; L^2 is additionally a *Hilbert space*, because its norm is given by an inner product, where $\langle f, g\rangle = \int fg$. (See next section for Hilbert space properties.)

Exercise. This is a different definition of L^1 than before. Can you prove that these definitions are equivalent?

3.1.3 L^p spaces are complete



$L^p = \{ f : f$ is measurable and $\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p} < \infty \}$

Example: $\forall n \in \mathbb{N}$, define $f_n(x) = e^{-x}$, where $0 \leq x \leq n$ (and $f = 0$ otherwise). Is f_n measurable for each n? What are the L^1 norms of f_n? Does f_n converge as a sequence of functions to a function **in** L^1 **norm** (not pointwise)? What is the limit, if so? Does this always work?

$$\{f_n(x)\}_{n=1}^{\infty} \xrightarrow{\text{Pointwise}} e^{-x}\Big|_{[0,\infty)}$$

$$\{\|f_n(x) - f\|_p\}_{n=1}^{\infty} \to 0$$

Properties of Norms

- $\|f\|_p \geq 0$, and $\|f\|_p = 0 \iff f \equiv 0$
- $\|cf\|_p = |c|\|f\|_p$
- $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

E.g. in L^1, $$\left\| e^{-x}\big|_{[0,n]} - e^{-x}\big|_{[0,\infty)} \right\|_1 = \int_0^{\infty} |f_n - f| d\mu \overset{MCT}{=} \int_n^{\infty} e^{-x} dx \xrightarrow{n\to\infty} 0$$

Lemma 3.1.2: Let $f_k \in L^p$ for $k = 1, 2, 3, \ldots$. If $\sum \|f_k\|_p$ converges as a real-valued series, then $\sum f_k$ converges to a function g in L^p. Moreover, $\sum f_k(x)$ converges to $g(x)$ pointwise for almost all x.

Hölder's Inequality

$$\int |fg| d\mu \leq \|f\|_p \|g\|_q \text{ for } \frac{1}{p} + \frac{1}{q} = 1$$

The Riesz-Fischer theorem: If $\{f_n\}$ is a Cauchy sequence of functions in an L^p space with $p \geq 1$ (so that, given $\varepsilon > 0$, there exists N so that $\|f_n - f_m\|_p < \varepsilon$ whenever $m, n \geq N$), then f_n converges to a function $f \in L^p$ in the L^p-norm limit.

Let $\{f_n\}$ be a Cauchy sequence. Let $\varepsilon = 2^{-k}$ for $k = 1, 2, \ldots$; we can find $N_k \in \mathbb{N}$ with $N_k < N_{k+1}$ such that

$$\|f_n - f_{N_k}\|_p < 2^{-k} \text{ for } n \geq N_k$$

Thus $\|f_{N_1}\|_p + \|f_{N_2} - f_{N_1}\| + \|f_{N_3} - f_{N_2}\| + \ldots$ is dominated by $\|f_{N_1}\|_p + \sum_{k=1}^{\infty} 2^{-k}$, thus converges strongly to f. Hence $\|f_{N_k} - f\|_p \to 0$, as the series telescopes.

Note then that $\|f_n - f\|_p = \|f_n - f_{N_k} + f_{N_k} - f\|_p \leq \|f_n - f_{N_k}\| + \|f_{N_k} - f\|_p = 2\varepsilon$, which converges for any $0 < \varepsilon < 2^{-k}$. $\blacksquare$

subject | date | keywords

topic

$f_n \to f$ weakly (in the weak topology) when

$\langle f_n, \varphi \rangle \to \langle f, \varphi \rangle$ for every $\varphi \in \mathcal{H}$

for any Hilbert space

$f_n \to f$ strongly (in the strong topology) when

$\|f_n - f\| \to 0$ for $f_n \in L^2$

Banach Space: Complete normed vector space

Hilbert Space: Vector space endowed with an inner product

Complete Space: Every Cauchy sequence converges to a limit w/i the space

subject | date | keywords

topic

Q 3.1.1.1

$$\|\cos x\|_{L^1(-\pi,\pi)} = \int_{-\pi}^{\pi} |\cos x|\,d\mu = 2\int_{0}^{\pi} |\cos x|\,d\mu = 2\left(\int_{0}^{\pi/2} \cos x\,d\mu - \int_{\pi/2}^{\pi} \cos x\,d\mu\right)$$

$$= 2\sin x\Big|_0^{\pi/2} - 2\sin x\Big|_{\pi/2}^{\pi} = \boxed{4}$$

Q 3.1.1.2

$$\|x^4\|_{L^2(0,1)} = \left(\int_0^1 |x^4|^2\,d\mu\right)^{1/2} = \left(\int_0^1 x^8\,d\mu\right)^{1/2} = \sqrt{\frac{x^9}{9}\Big|_0^1} = \sqrt{\frac{1}{9}} = \boxed{\frac{1}{3}}$$

E 3.1.9

Show $f_n(x) = x^n$ converges strongly in $L^2(0,1)$ to $f(x) = 0$

$$\lim_{n\to\infty} \|f_n - f\|_2 = \lim_{n\to\infty} \|x^n - 0\| = \lim_{n\to\infty} \left(\int_0^1 x^{2n}\,d\mu\right)^{1/2} = \lim_{n\to\infty} \left(\frac{1}{2n+1}\right)^{1/2} = 0 \checkmark$$

Q 3.1.1.6

Find the $L^2(0,4)$ distance between $1+x^2$ and $2\sqrt{x}$

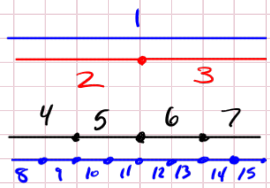
$$\|1+x^2-2\sqrt{x}\|_2 = \left(\int_0^4 |1+x^2-2\sqrt{x}|^2\,d\mu\right)^{1/2}$$

$$= \left(\int_0^4 \left(1+2x^2+x^4-4\sqrt{x}-4x^{5/2}+4x\right)d\mu\right)^{1/2}$$

$$= \sqrt{\frac{12164}{105}} \approx 10.76$$

Q 3.1.c

$$\chi_{[0,1]},\ \chi_{[0,1/2]},\ \chi_{[1/2,1]},\ \chi_{[0,1/4]},\ \chi_{[1/4,1/2]},\ \chi_{[1/2,3/4]},\ \chi_{[3/4,1]},\ \dots$$



$$\left\|\chi_{[\frac{k}{2^n},\frac{k+1}{2^n}]} - 0\right\|_2 = \int_0^1 \chi^2_{[\frac{k}{2^n},\frac{k+1}{2^n}]}\,d\mu = \int_{k/2^n}^{(k+1)/2^n} d\mu = \frac{1}{2^n} \to 0$$

Let $x_0 \in [0,1]$

$\lim_{n\to\infty} \chi_n(x_0)$ DNE

Pointwise non convergence

$$\{\underline{1}, \underline{0,1}, \underline{0,0,1,0}, \underline{0,0,0,0,0,0,1,0}, \dots\}$$

3.2 Hilbert Space Properties of L^2 and ℓ^2

Key Concepts

Definition. The inner product of two functions f and g in $L^2(a,b)$ is $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$. (The inner product on $L^2(\mathbb{R})$ is $\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx$.)

Having *any* kind of inner product gives a Hilbert space quite a few interesting properties:

Properties of the Inner Product on any (real) Hilbert Space

For vectors $f, g, h \in \mathcal{H}$ and real constants a and b:

1. $\langle f,f\rangle = \|f\|^2$
2. $\langle f,f\rangle \geq 0$ and equals 0 only when $f = 0$
3. $\langle f,g\rangle = \langle g,f\rangle$
4. $\langle af + bg, h\rangle = a\langle f,h\rangle + b\langle g,h\rangle$
5. (Schwarz Inequality) $|\langle f,g\rangle| \leq \|f\| \cdot \|g\|$
6. (Angle between vectors) $\cos\theta = \dfrac{\langle f,g\rangle}{\|f\|\|g\|}$
7. (The Pythagorean Theorem) If two L^2 functions f and g are at right angles, then $\|f\|^2 + \|g\|^2 = \|f+g\|^2$
8. (Parallelogram Law) $\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$

If we allow for complex functions let $f(t) = \frac{1}{\sqrt{2\pi}} e^{int}$ for $n \in \mathbb{Z}$

$L^2(-\pi,\pi)$: $\|f\|_2^2 = \int_{-\pi}^{\pi} \left|\frac{e^{int}}{\sqrt{2\pi}}\right|^2 dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int} e^{-int}\, dt = 1$

Sps $g(t) = \frac{1}{\sqrt{2\pi}} e^{imt}$ for $m \neq n$

$$\langle f,g\rangle = \int_{-\pi}^{\pi} \frac{e^{int}}{\sqrt{2\pi}} \cdot \frac{e^{-imt}}{\sqrt{2\pi}}\, dt = \frac{e^{i(n-m)t}}{2\pi i(n-m)}\Bigg|_{-\pi}^{\pi} = \frac{1}{2\pi i(n-m)}\left(e^{i\pi(n-m)} - e^{-i\pi(n-m)}\right)$$

$$= \frac{1}{2\pi i(n-m)}(0) = 0$$

Orthonormal Basis - see Fourier series

subject | date | keywords

topic

Compute $\|1-\cos x\|_1$ in $L^1(-\pi,\pi)$

$$= \int_{-\pi}^{\pi} |1-\cos x|\,dx = \int_{-\pi}^{\pi} (1-\cos x)\,dx = x-\sin x\Big|_{-\pi}^{\pi} = 2\pi$$

$\|1-\cos x\|_2$ in $L^2(-\pi,\pi)$

$$=\left(\int_{-\pi}^{\pi} |1-\cos x|^2\,dx\right)^{1/2} = \left(\int_{-\pi}^{\pi} 1-2\cos x+\cos^2 x\,dx\right)^{1/2}$$

$$=\left(\int_{-\pi}^{\pi} 1-2\cos x+\tfrac{1}{2}(1+\cos(2x))\,dx\right)^{1/2}$$

$$=(3\pi)^{1/2} = \sqrt{3\pi}$$

Exercises

1. Find the inner product of $f(x) = 3x^2 + 1$ and $g(x) = 2x$ on $L^2(1,3)$. Also find the angle between the vectors. What linear function in $L^2(1,3)$ is perpendicular to f?

2. Prove the Schwarz inequality for an arbitrary L^2 space.

3. Use inner product properties 1-4 to prove the Pythagorean Theorem and the Parallelogram Law.

4. The Hilbert space ℓ^2 is the collection of infinite sequences (vectors) $v = (a_0, a_1, a_2, ...)$ such that $\|\vec{v}\|^2 = \sum_{j=0}^{\infty} |a_j|^2 < \infty$. ℓ^2 is a *complex* Hilbert space, meaning that vector entries can be complex numbers. The inner product is then conjugate-linear in the second coordinate: if $v = (a_0, a_1, a_2, ...)$ and $w = (b_0, b_1, b_2, ...)$, then $\langle v, w\rangle = \sum_{j=0}^{\infty} a_j \overline{b_j}$.

   Prove that for complex α with $|\alpha| < 1$, that $\vec{K_\alpha} = (1, \alpha, \alpha^2, \alpha^3, ...)$ belongs to ℓ^2 and compute its norm.

3.3 Orthonormal Basis for Hilbert Space

Key Concepts

1. A *orthonormal basis* for a Hilbert space
2. The *Gram-Schmidt* process for generating an orthonormal basis
3. The *Laguerre polynomials* that give an orthonormal basis for L^2
4. The *Projection Theorem* that allows a Hilbert space to be decomposed into subspaces
5. An *isometric isomorphism* between Hilbert spaces to show they are equivalent
6. *Parseval's identity* established by the isometric isomorphism between L^2 and ℓ^2

3.3.1 A Vector Space Basis

Inquiry. How do you break a vector in $\mathbb{R}^3$ into "pieces"? How could I do the same with a vector in ℓ^2?

Definition. A set of elements $\{b_n\}$ in a vector space equipped with an inner product is **orthogonal** when $\langle b_m, b_n \rangle = 0$ for two distinct elements b_m and b_n. An orthogonal **basis** for a Hilbert space $\mathcal{H}$ is a set of orthogonal elements $\{b_n\}$, where each $b_n \in \mathcal{H}$ has the property

For $f \in \mathcal{H}$, if $\langle f, b_n \rangle = 0$ for every vector b_n in the set, then $f = 0$.

The basis is **orthonormal** if, in addition, each basis element has $\langle b_n, b_n \rangle = 1$. The (well-defined) dimension of the Hilbert space is the cardinality of any of its orthogonal basis sets.

Inquiry. How would you suggest we find an orthonormal basis for L^2?

3.3.2 The Gram-Schmidt Process

Gram-Schmidt Process. The *Gram-Schmidt* process consists of the following algorithm performed on a given finite set of vectors $\{g_0, \ldots g_n\}$:

$$\text{Start by letting } f_0 = g_0.$$

$$\text{Then set } f_1 = g_1 - \frac{\langle g_1, f_0 \rangle}{||f_0||^2} f_0.$$

$$\text{Next, } f_2 = g_2 - \frac{\langle g_2, f_1 \rangle}{||f_1||^2} f_1 - \frac{\langle g_2, f_0 \rangle}{\|f_0\|^2} f_0.$$

Continue this process until $f_0, \ldots, f_n$ have been defined (the same number of functions as the number of separate functions g in the original set). At each step, use the rule

$$f_j = g_j - \frac{\langle g_j, f_{j-1} \rangle}{\|f_{j-1}||^2} f_{j-1} - \frac{\langle g_j, f_{j-2} \rangle}{||f_{j-2}\|^2} f_{j-2} - \ldots - \frac{\langle g_j, f_0 \rangle}{\|f_0\|^2} f_0.$$

Exercise. Prove that the Gram-Schmidt process gives a set of mutually orthogonal vectors.

Exercise. Use the Gram-Schmidt process on the collection of $L^2(0,\infty)$ functions $\{g_0, g_1, g_2, g_3\}$ where $g_n(x) = x^n e^{-x/2}$ to generate a set of *orthonormal* functions $\{e_0, e_1, e_2, e_3\}$. These are (weighted) Laguerre polynomials.

The Gram-Schmidt process can continue indefinitely, but not does not always create a basis when applied to a countably infinite collection of functions. The Laguerre polynomials *do* form a basis for $L^2(0,\infty)$ and can be defined directly as

$$e^{-x}L_n(x) = \tfrac{1}{n!}(\tfrac{d}{dx})^n(x^n e^{-x}),\ n = 0,1,2,\ldots,$$

Exercise. Show that the Laguerre (*unweighted*) polynomial e_1 obtained from the previous exercise matches this definition up to a scalar constant. In other words, show $e_1 e^{x/2} = c_1 L_1(x)$. Then do the same for e_2 and $L_2(x)$.

3.3.3 The Projection Theorem

Definition. The linear *span* of a collection of functions $g_0, g_1, \ldots, g_n$ is the subspace of a vector space given by every linear combination of those elements, i.e.

$$\text{span}\{g_0, g_1, \ldots, g_n\} = \{a_0 g_0 + \ldots + a_n g_n : a_1, \ldots, a_n \in \mathbb{R}\}.$$

Best Approximation. Given a function f in L^2, there exists a best approximation to f that is in the (closed) linear span, M, of a collection of functions $g_0, g_1, \ldots, g_n$. The best approximation to f by an element g in M is the one that minimizes the value of $\|f - g\|$, and this is given by

$$g = c_0 f_0 + c_1 f_1 + \ldots + c_n f_n, \text{ where } c_j = \frac{\langle f, f_j\rangle}{\|f_j\|^2}, \text{ and where the functions } f_0, \ldots, f_n$$
are produced when the Gram-Schmidt process is applied to $\{g_0, g_1, \ldots, g_n\}$.

This also means that $\langle f - g, h\rangle = 0$ for every $h \in M$. The set of elements in the vector space that, like $f - g$, are orthogonal to all elements of M, are denoted $M^\perp$. The best approximation g is the **orthogonal projection** of f onto M.

Exercise. The function $f(x) = x^2, 0 < x < 1$, is in $L^2(0,\infty)$. Define a subspace
$M = \{a_0 g_0 + a_1 g_1 + a_2 g_2 :\ a_0, a_1, a_2 \in \mathbb{R}\}$ with $g_n(x) = x^n e^{-x/2}$, $n = 0, 1, 2$. We determine the best approximation to $f(x)$ in M and write $f = g + h$, where $g \in M$ and $h \in M^\perp$.

The Projection Theorem. If M is a closed linear subspace of L^2, then $f \in L^2$ can be uniquely expressed as $f = g + h$, where $g \in M$ and $h \in M^\perp$.

Exercise. Prove the Projection Theorem.

Exercise. In $L^2(0,1)$, define $M = \{g(x) = c_0 + c_1 x\}$, where $c_0, c_1 \in \mathbb{R}$. Use the projection theorem and the Gram-Schmidt process to describe $M^\perp$.

3.3.4 Equivalence of Hilbert Spaces

Sometimes, there is a complete identification between two Hilbert Spaces, similar to a group isomorphism or a continuous bijection between topological spaces.

Definition. A Hilbert space **isomorphism** is a one-to-one linear map T from one Hilbert space $\mathcal{H}$ onto another $\mathcal{K}$ that satisfies $\|T(h)\|_{\mathcal{K}} = \|h\|_{\mathcal{H}}$. When such an isomorphism T exists, we say $\mathcal{H}$ and $\mathcal{K}$ are isometrically isomorphic to one another.

Exercise. Prove that a Hilbert space isomorphism T from a Hilbert space $\mathcal{H}$ to another $\mathcal{K}$ satisfies

$$\langle T(g), T(h)\rangle_{\mathcal{K}} = \langle g, h\rangle_{\mathcal{H}}, \text{ for } g, h \in \mathcal{H}.$$

Exercise. Use the (weighted) Laguerre polynomials to construct a Hilbert space isomorphism between $L^2(0, \infty)$ and ℓ^2, and prove that they are isomorphic.

Sample Test Problems for Chapter 3

1. Find the angle between $1 - x$ and x^2 in $L^2[0,1]$.

2. Prove that the sequence of vectors α_n in l^2, where $\alpha_n = (1, \alpha, \alpha^2, ..., \alpha^n, 0, 0, 0....)$ for some real $\alpha, 0 < \alpha < 1$, is Cauchy in l^2.

3. Find the L^2-norm limit of the sequence $f_n(x) = \frac{1}{2^n}x + 1 - \frac{1}{2^n}$ and prove your answer.

4. Show that if equality is attained in the Cauchy-Schwarz inequality for two vectors f, g, then they are linearly dependent.

5. Use the Gram-Schmidt process to convert the set $\{1, x, x^2\}$ into three mutually orthogonal vectors on $L^2[0,1]$.

6. Prove that if $f \in L^2[0,1]$ and $g \in L^3[0,1]$, then $fg \in L^{6/5}[0,1]$.

7. Let $k \in \mathbb{N}$ be fixed and given. Prove the function sequence $f_n(x) = x^k/n$, where $n = 1, 2, 3, \ldots$, converges to $f(x) = 0$ in the $L^2(0,1)$ norm.

8. Prove the **Riesz-Fischer Theorem:** If $\{f_n\}$ is a Cauchy sequence of functions in an L^p space with $p \geq 1$ (so that, given $\varepsilon > 0$, there exists N so that $\|f_n - f_m\|_p < \varepsilon$ whenever $m, n \geq N$), then f_n converges to a function $f \in L^p$ in the L^p-norm limit.
   In your proof, feel free to use Lemma 3.1.2: Let $f_k \in L^p$ for $k = 1, 2, 3, \ldots$. If $\sum \|f_k\|$ converges as a real-valued series, then $\sum f_k$ converges to a function g in L^p. Moreover, $\sum f_k(x)$ converges to $g(x)$ pointwise for almost all x.

9. Prove the Pythagorean theorem holds in $L^2(0,1)$ for the functions $f(x) = x^2$ and $g(x) = 3x^2 - 9/5$. Exhibit the values for $\|f\|^2$, $\|g\|^2$, and $\|f + g\|^2$.

10. For $f(x) = 1$ and $g(x) = \sqrt{x-1}$, find the values of the inner product $\langle f, g \rangle$ and the norms $\|f\|$ and $\|g\|$ in $L^2(1,5)$. Then determine an approximate radian measurement for the angle between f and g, as the functions are thought of as elements of $L^2(1,5)$.

11. a. Show, for complex z with $|z| < 1$, the vector $\vec{k_z} = \begin{bmatrix} 1 \\ z \\ z^2 \\ z^3 \\ \vdots \end{bmatrix}$ is an element of ℓ^2. b. Did you love Taylor series from Calculus 2? (Answer yes or no, and give a simple reason why.) Then suppose $\vec{f} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} \in \ell^2$. Find an infinite series representation for its ℓ^2 inner product with $\vec{k_z} = \begin{bmatrix} 1 \\ z \\ z^2 \\ z^3 \\ \vdots \end{bmatrix}$, where $z \in \mathbb{R}$ has $|z| < 1$.

12. a. Give simple reasons why:

    i. The function sequence $f_n(x) = \mathcal{X}_{[0,n]}(x)\mathrm{e}^x$, where $n = 1, 2, 3, \ldots$, converges pointwise at every $x \in [0, \infty)$ to $f(x) = \mathrm{e}^x$.

    ii. Each element of that function sequence is in $L^1(0, \infty)$.

    iii. The function sequence does not converge to $f(x)$ in the $L^1(0, \infty)$ norm.

    b. From this example, what can you conclude about pointwise convergence and strong convergence?

13. (Take-Home problem?) Find a sequence of functions that converges pointwise $\forall x \in [0, 1]$ but does not converge in $L^2[0, 1]$-norm.

Chapter 4

Measure Theory

4.1 Lebesgue Measure

Key Concepts

1. Definition of *Lebesgue measurable* set
2. Properties of *Lebesgue measure*
3. Other definitions of *measure*
4. Other examples of *measure*

Definition. A set S in $\mathbb{R}$ is *Lebesgue measurable* when its characteristic function $\mathcal{X}_S$ is a measurable function. In this case, we define *the Lebesgue measure of* S, denoted $m(S)$, as

$$m(S) = \int \mathcal{X}_S$$

whenever $\mathcal{X}_S \in L^1$. If $\mathcal{X}_S$ is not Lebesgue integrable, then we define $m(S) = \infty$.

Examples. Let $\emptyset$ be the empty set, C be the Cantor set and V the Vitali set.

1. $m([a, b]) = b - a$
2. $m(\emptyset) = 0$
3. $m(C) = 0$
4. $m(V) = \infty$

4.1.1 Properties of Lebesgue measure

Although not every set is measurable, the following theorems show that measurable sets can be combined through union and intersection.

Theorem. If S and T are Lebesgue measurable sets in $\mathbb{R}$, then so are $S \cup T$ and $S \cap T$, and

$$m(S \cup T) = m(S) + m(T) - m(S \cap T).$$

Theorem. Suppose $S = \bigcup_{n=1}^{\infty} S_n$, where S_n is a measurable set. Then S is measurable with

$$m(S) \leq \sum_{n=1}^{\infty} m(S_n).$$

If in addition the sets S_n are disjoint, so $S_j \cap S_k = \emptyset$ when $j \neq k$, then

$$m(S) = \sum_{n=1}^{\infty} m(S_n).$$

Inquiry. In Section 1.2, we said that $m(S) = 0$ when S can be covered with a sequence of open intervals $I_1, I_2, I_3, \ldots$ whose bounded total measure $\sum_{n=1}^{\infty} m(I_n)$ is arbitrarily small. Are these definitions equivalent? Can you prove it?

4.1.2 Other Ways to Define Measure

The Lebesgue measure is not the only possible measure on sets. Here are listed of desired properties for a measure.

1. *Universality*: the measure of any real set to be properly defined.
2. *Countable Additivity*: the measure of the union of a countable number of disjoint sets to equal the summation of the measure of each.
3. *Length Agreement*: for any interval I, the measure of I to equal its length.
4. *Translation Invariance*: the measure of any set E to equal, for any given real c, the measure of the translated set $E + c = \{x + c : x \in E\}$.

Lebesgue measure accomplishes 2, 3, 4, but not 1 (the Vitali set is not measurable).

Example A unit point mass measure m_p is defined for a real number (the "point") p. For a set S, if $p \in S$, then $m_p(S) = 1$, and if $p \notin S$, then $m_p(S) = 0$. The measure of every set S is defined, and so m_p satisfies the property of universality.

For example, let $p = 1/2$. Depending upon whether or not $1/2$ is in the set, its point mass measure at $1/2$ is either 0 or 1. So $m_{1/2}(\mathbb{R}) = 1$, $m_{1/2}([0, 1]) = 1$, $m_{1/2}([-10, 10]) = 1$, $m_{1/2}([3, 4]) = 0$, and $m_{1/2}(C) = 0$ for the Cantor set C because $1/2 \notin C$. You can see $m_{1/2}$ does not satisfy the property of length agreement.

Question. Is an arbitrary point mass measure m_p countably additive? Is it translation invariant?

Remark. No measure can satisfy all four properties! Proof as follows:

1. Assume all four properties apply to an arbitrary measure m.
2. Define a relation on $[0, 1)$: a and b in $[0, 1)$ are related if $a-b$ is a rational number. (E.g., all rational numbers in $[0, 1)$ are related.)
3. Choose one number from each equivalence class and put them together into a set S.
4. Enumerate the rationals in $[0, 1)$: $\{0, r_1, r_2, r_3, \ldots\}$.
5. Consider the sets $S_n = \{s + r_n : s \in S\}$, $n = 0, 1, 2, \ldots$.

   (Notice $S = S_0$, and the S_n are mutually disjoint.)
6. Note $[0, 1) = \bigcup_{n=0}^{\infty} S_n$. So $m([0, 1)) = \sum_{n=0}^{\infty} m(S_n)$.
7. Note $m(S_n) = m(S)$ for each of the sets S_n, by translation invariance.
8. What's the problem?

Inquiry. Get in three groups and invent your own measures! One group should find a measure that satisfies principles 1, 3, 4, and not 2. The next group, find a measure that satisfies 1, 2, 4, and not 3. The last group, find a measure that satisfies 1, 2, 3 and not 4.

4.2 Integrals with Respect to Other Borel Measures.

Key Concepts

1. A *σ-algebra* of *Borel sets*
2. A *Borel measure* on a *σ-algebra* of *Borel sets*
3. More examples of *Borel measures*
4. The *Lebesgue integral* with respect to a *Borel measure*

Definitions.

1. First, form a "**σ-algebra of Borel sets** $\mathcal{B}$":

   - Start by putting all the intervals of $\mathbb{R}$ (including $\mathbb{R}$, single-point intervals of the form $\{x\}$, other infinite intervals, and the empty set) into $\mathcal{B}$.
   - Then add into $\mathcal{B}$ all of the sets formed as countable unions, or countable intersections, or complements of these intervals.
   - Then add in all of the sets that are countable unions, intersections, or complements of anything already in $\mathcal{B}$.
   - Continue, forming $\mathcal{B}$ as closed with respect to countable unions, countable intersections, and complements.

   In this way, a σ-algebra is a collection of sets $\mathcal{B}$ where every union or intersection of a countable collection of sets in $\mathcal{B}$, along with any complement, is again in $\mathcal{B}$.

2. Now you can define a general (not just Lebesgue) measure μ on each set in the σ-algebra $\mathcal{B}$. We call such measures a Borel measure. A measure is technically a function $\mu : \mathcal{B} \to [0, \infty]$, defined on all of $\mathcal{B}$, with three main properties.

   (a) It is defined for each interval of $\mathbb{R}$, with the measure of the empty set $\mu(\emptyset)$ being zero and (so the measure is not trivial or always infinite) at least one interval having finite nonzero measure.

   (b) It is countably additive, so $\mu(\cup B_n) = \sum \mu(B_n)$ for a countable number of disjoint sets B_n. (The fact that we are working inside a σ-algebra assures us—whenever each B_n is a member of $\mathcal{B}$—that $\cup B_n$ is also a member of $\mathcal{B}$.)

   (c) We insist μ is defined so it is finite on any closed and bounded set, which means μ is defined so it is finite on every closed and bounded interval.

3. **The Lebesgue Integral with Respect to a Borel Measure μ:** Take a Borel measure μ, which has a measure $\mu(I)$ for any interval I. Then:

   (i) The integral with respect to μ of the step function $\phi(x) = \sum_{j=1}^{n} c_j \cdot \mathcal{X}_{I_j}(x)$ is
   $$\int_{-\infty}^{\infty} \phi(x)\, d\mu(x) = \sum_{j=1}^{n} c_j \cdot \mu(I_j).$$
   This integral is also denoted $\int \phi\, d\mu$.

   (ii) If a nondecreasing sequence of step functions $\{\phi_n\}$ converges μ-almost everywhere (except possibly on a set S with $\mu(S) = 0$) to a function f, then the integral of f with respect to μ is
   $$\int f\, d\mu = \lim_{n\to\infty} \int \phi_n(x)\, d\mu(x).$$
   If the integral is finite, then f is said to be in the collection of functions $L^0(\mu)$.

   (iii) Whenever a function f can be written as $f = g - h$, where $g, h \in L^0(\mu)$, then
   $$\int f\, d\mu = \int g\, d\mu - \int h\, d\mu.$$
   Such functions f are said to be in the function space $L^1(\mu)$.

Example. Point Mass Measure: First define μ on an interval like this:

Pick a real number k as a "weight." Then $\mu_p(I) = k$ for any interval I that contains a given point p, and $\mu_p(I) = 0$ for an interval I not containing p. This measure turns out to produce $\mu_p(S) = 0$ precisely when S does not contain p. The measure generates an integral of a function f with respect to μ_p that, essentially, evaluates f at p and then multiplies by k. For if $\phi = \sum_{j=1}^{n} c_j \cdot \mathcal{X}_{I_j}(x)$ is a step function (without loss of generality, we may assume the intervals I_j are disjoint), then $\int_{-\infty}^{\infty} \phi(x)\ d\mu(x) = \sum_{j=1}^{n} c_j \cdot \mu_p(I_j) = \sum_{\{j:p\in I_j\}} c_j \cdot k = \sum_{j=1}^{n} c_j \cdot k \cdot \mathcal{X}_{I_j}(p) = k \cdot \phi(p)$. Working through the rest of the three-step process, we get $\int_{-\infty}^{\infty} f(x)\ d\mu_p(x) = k \cdot f(p)$ for $f \in L^1(\mu_p)$.

Problems.

1. For an interval I, calculate $\int \mathcal{X}_I\ d\mu$, where μ is the point mass $\mu = 3\mu_{1/2}$. Hence, for an interval I, $\mu(I) = 3\mu_{1/2}(I) = 3$ if $1/2 \in I$, and $\mu(I) = 0$ if $1/2 \notin I$.

2. What is $\int \mathcal{X}_{[0,2)\cup(5,12)}\ d\mu$?

3. What is $\int \mathcal{X}_C\ d\mu$, where C is the Cantor set?

4. For a real-valued function f whose domain includes $x = 1/2$, calculate $\int f d\mu$.

Example. Absolutely Continuous Measure. An *absolutely continuous measure* (with respect to Lebesgue measure), which we call ν, produces an integral of the form $\int g\, d\nu = \int g(x) \cdot w(x)\, dx$, where the function w is measurable and takes on only nonnegative values, and dx refers to Lebesgue measure. The function w is sometimes called a weight function or a density function. It turns out $\nu(S) = 0$ exactly when S has Lebesgue measure zero. Here's an example, where we use the three-step process to construct the integral with respect to a measure ν.

We choose $w(x) = \mathrm{e}^{-x^2}/\sqrt{\pi}$, and develop the corresponding integral. Before proceeding, we verify w is in $L^2(\mathbb{R})$ (w is measurable because it is continuous). The square of the L^2 norm is

$$\|w\|_2^2 = \int_{-\infty}^{\infty} \mathrm{e}^{-2x^2}/\pi\, dx = \int_{-\infty}^{\infty} \mathrm{e}^{-u^2}/(\sqrt{2}\pi)\, du = 1/\sqrt{2\pi}.$$

Use w to define the measure of an interval $I = (a, b)$, where a and b can be allowed to equal ∞ or $-\infty$:

$$\nu(I) \equiv \int_{-\infty}^{\infty} \mathcal{X}_I(x) \cdot \mathrm{e}^{-x^2}/\sqrt{\pi}\, dx.$$

Any absolutely continuous measure having a given weight function w works this way. I

First, for a step function $\phi(x) = \sum_{j=1}^{n} c_j \cdot \mathcal{X}_{I_j}(x)$, we have $\int \phi\, d\nu = \sum_{j=1}^{n} c_j \cdot \nu(I_j)$.

Applying the formula for $\nu(I)$, $\int \phi\, d\nu = \sum_{j=1}^{n} c_j \cdot [\int_{-\infty}^{\infty} \mathcal{X}_{I_j}(x) \cdot \mathrm{e}^{-x^2}/\sqrt{\pi}\, dx]$.

Interchanging the finite sum and the integral sign, $\int \phi\, d\nu = \int_{-\infty}^{\infty} (\sum_{j=1}^{n} c_j \cdot \mathcal{X}_{I_j}(x)) \cdot \mathrm{e}^{-x^2}/\sqrt{\pi}\, dx = \int_{-\infty}^{\infty} \phi(x) \cdot \mathrm{e}^{-x^2}/\sqrt{\pi}\, dx$.

The main point is that the integral of the step function with respect to ν turns out to be the Lebesgue integral of the step function multiplied against the weight function w. This integral exists (so it is well-defined), because $\int_{-\infty}^{\infty} \phi(x) \cdot \mathrm{e}^{-x^2}/\sqrt{\pi}\, dx \leq \int_{-\infty}^{\infty} \phi(x)/\sqrt{\pi}\, dx < \infty$ and the function $\phi(x) \cdot \mathrm{e}^{-x^2}/\sqrt{\pi}$ is continuous almost everywhere.

Second, given a nondecreasing sequence of step functions ϕ_n that converges a.e. (in the sense of ν, which is the same as in the sense of Lebesgue measure) to a function f, the integral of f is

$$\int f\, d\nu = \lim_{n\to\infty} \int \phi_n\, d\nu.$$

By the monotone convergence theorem,

$$\int f\, d\nu = \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x) \cdot \mathrm{e}^{-x^2}/\sqrt{\pi}\, dx = \int_{-\infty}^{\infty} \lim_{n\to\infty} \phi_n(x) \cdot \mathrm{e}^{-x^2}/\sqrt{\pi}\, dx = \int f(x) \cdot \mathrm{e}^{-x^2}/\sqrt{\pi}\, dx.$$

In short, the integral is understood as the Lebesgue integral of f multiplied by the weight $\mathrm{e}^{-x^2}/\sqrt{\pi}$. We say such functions f, producing a finite integral, are in $L^0(\nu)$.

Third, when f can be written as $f = g - h$, for g and h in $L^0(\nu)$, then $f \in L^1(\nu)$ and $\int f\, d\nu = \int g\, d\nu - \int h\, d\nu$. From the last paragraph, we see $\int f\, d\nu = \int f(x) \cdot \mathrm{e}^{-x^2}/\sqrt{\pi}\, dx$ for such functions.

Problem. Using the weight function $w(x) = \mathrm{e}^{-x^2}/\sqrt{\pi}$ to form the absolutely continuous measure ν so that $d\nu(x) = w(x)dx$, find $\int x\mathcal{X}_{(0,\infty)}(x)\, d\nu(x)$ and $\int x^2\, d\nu(x)$.

Example. Singular Continuous Measure: The integration process also works for singular continuous measures, constructed from a function F that is **singular** with respect to Lebesgue measure. That means $F'(x) = 0$ a.e. and F is continuous. The associated Borel measure μ_F is defined as $\mu_F([a, b]) = F(a) - F(b)$ on any interval I with endpoints a and b, and the interval can be open or half-open as well.

An example is the famous *Cantor function* F, which is constant on the complement of the Cantor set C. The function's growth is supported on $[0, 1]$, equals 0 for negative x values, and equals 1 for $x > 1$. For $x \in [0, 1]$, write x in its ternary (base 3) expansion. For technical reasons that make this definition work, do not use a 1, when possible, in the expansion (for example, when representing $x = 1/3$ we choose $0.02222\ldots[3]$ instead of $0.10000.\ldots[3]$). Next, replace the first 1 in the expansion with a 2 and everything after it with 0. Finally, replace any 2 in the resulting expansion with a 1, and interpret the result as a binary number. That number is $F(x)$.

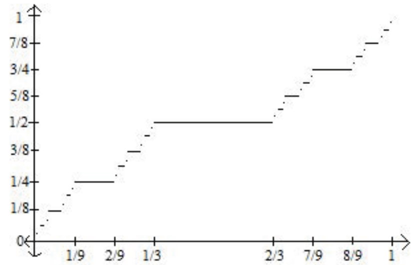


Figure 4.1: The Cantor Function is often fancifully called the *Devil's staircase*.

Calculating F. For example, $1/2 = 0.111111\ldots[3]$ in base 3 (there is no way not to use 1s in the expansion).

We replace the first 1 with a 2 and everything after it with a 0, obtaining $0.2000\ldots$.

Now we replace the 2 with a 1, obtaining $0.1000....$

Interpreting that expansion in base 2, we get the resulting function value: $F(1/2) = 1/2 + 0/2^2 + 0/2^3 + \cdots = 1/2$.

Many values are easy to pick out of the graph visually; e.g., $F(1/3) = 1/2,\ F(1/9) = 1/4$, and $F(7/9) = 3/4$.

The associated measure on intervals follows. For example, $\mu_F([2/9, 1/3)) = F(1/3) - F(2/9) = 1/2 - 1/4 = 1/4$, $\mu_F([1/3, 2/3]) = 1/2 - 1/2 = 0,$ and $\mu_F((2/3, 7/9) = 3/4 - 1/2 = 1/4$.

Exercise. Find $\int x\, d\mu_F(x)$. (*Hint:* This shows how singular continuous measures can form challenging problems. See Exercise 4.2.25.)

4.3 $L^2(\mu)$

The Key Concept. A function is *μ-measurable* when $\text{mid}\{-g, f, g\}$ is in $L^1(\mu)$ for every nonnegative $g \in L^1(\mu)$. The *Hilbert space* $L^2(\mu)$ is then the collection of μ-measurable functions f with finite $L^2(\mu)$ norm, which is defined as $\|f\| = \sqrt{\int |f|^2 \, d\mu}$.

Exercises.

1. For a function f whose domain includes $x = 1/2$ and for the point-mass measure $\mu_{1/2}$, define the $L^2(\mu)$ norm of f to be $\left(\int |f|^2 \, d\mu \right)^{1/2}$. Calculate the $L^2(\mu)$ norm of $f(x) = x$.

2. Describe the collection of functions in $L^2(\mu)$ when μ is defined as the sum of two unit point mass measures $\mu = \mu_{1/2} + \mu_0$. Here, $\mu_{1/2}$ is the unit point mass measure at $p = 1/2$, so for a real set S, $\mu_{1/2}(S) = 1$ if $1/2 \in S$ and $\mu_{1/2}(S) = 0$ if $1/2 \notin S$. The point mass μ_0 is defined the same way, except the point mass is at $p = 0$.

   (a) What real-valued functions are in $L^2(\mu)$? (*Hint:* such a function must be well-defined at 0.)

   (b) Why is the space $L^2(\mu)$ two-dimensional? (*Hint:* describe it as the space of linear functions $f(x) = c_1(1/2 - x) + c_2 x$.)

   (c) Why do the functions $1 - 2x$ and $2x$ form an orthonormal basis for $L^2(\mu)$? (Note the normalizations of the elements.)

   (d) How can you describe any function f in $L^2(\mu)$ in a Fourier series expansion of the form $f = \langle f(x), 1 - 2x \rangle \cdot (1 - 2x) + \langle f(x), 2x \rangle \cdot 2x$?

4.4 Probability

Key Concepts

1. An *experiment* is a general term describing an activity that can result in many different outcomes,
2. A *sample space* of an experiment is set of all possible outcomes. Many times because of simple probability formulas that are especially helpful when the sample space is finite, probabilists try to express the sample space as "equiprobable," where all the outcomes have an equal probability.
3. An *outcome* of an experiment is the most fundamental type of result of the experiment. An *event* is a collection of outcomes—a subset of the sample space.
4. A *random variable* assigns a real number to each outcome of an experiment.
5. A *probability space* $(S, \mathbb{F}, \mu)$ is a trio that models (describes) probabilistic behavior. Here, S is the sample space and $\mathbb{F}$ represents the set of all subsets of S, where $\mathbb{F}$ is required to form a σ-algebra of subsets. μ is a probability distribution—a probability measure.

Definitions. 1. Given an experiment and a sample space S of all possible outcomes, a random variable X assigns a real number x to each outcome $s \in S$. We sometimes write $X(s) = x$. A real-valued random variable X with Borel distribution is a random variable defined on a probability space $(S, \mathbb{F}, \mu)$, where $S \subseteq \mathbb{R}$ is the sample space of values X assigns. $\mathbb{F}$ is required to be a collection of sets containing elements of S (always including the empty set and S itself) and for which any complement of a set in $\mathbb{F}$ is again in $\mathbb{F}$ and every countable union of sets in $\mathbb{F}$ is again in $\mathbb{F}$. Furthermore, we require μ to be a Borel measure in the sense of Section 4.2, for which every set in $\mathbb{F}$ is μ-measurable and for which $\int \mathcal{X}_S \, d\mu = 1$.

2. For such a real-valued random X, the probability that X assigns a number in a set A of $\mathbb{F}$ is $P[X \in A] = \int \mathcal{X}_A \, d\mu$.

Examples.

1. Suppose $X =$ the roll of a fair die. Then $P[X \leq x] = x/6$ for $x = 1, 2, 3, 4, 5, 6$.

2. Suppose R is the total on a roll of two fair six-sided dice, where . The two outcomes, for example, of *rolling a three*, which are "1, 2" and "2 , 1" (both have $R = 3$) are different. There are $6 \cdot 6 = 36$ two-dice rolls. The sample space is $S = \left\{ \begin{smallmatrix} 11, & 12, & 13, & 14, & 15, & 16 \\ 21, & 22, & 23, & 24, & 25, & 26 \\ 31, & 32, & 33, & 34, & 35, & 36 \\ 41, & 42, & 43, & 44, & 45, & 46 \\ 51, & 52, & 53, & 54, & 55, & 56 \\ 61, & 62, & 63, & 64, & 65, & 66 \end{smallmatrix} \right\}$ $\mathbb{F}$ is the collection of all the 2^{36} subsets of S, and μ satisfies $\mu(\{s\}) = 1/36$ for any $s \in S$, and (for any general $A \subset \mathbb{F}$) $\mu(A) = n/36$, where $|A| = n$. Therefore, for example, $\mu(\{R = 7\} = \mu(\{16, 25, 34, 43, 52, 61\}) = 6/36 = 1/5$.

3. Suppose U is a continuous random variable (one that has an absolutely continuous probability measure) that assigns values x in the set $[0, 1]$ with equal weight, so the density function is $f(u) = \mathcal{X}_{[0,1]}(u)$. ($U$ is the uniform random variable on $[0, 1]$.) Then the sample space is $S = [0, 1]$, $\mathbb{F}$ consists of all the Lebesgue measurable subsets of $[0, 1]$, and, for $A \subset \mathbb{F}$, $\mu(A) = m(A)$.

Exercises.

1. Let Y be the number of heads tossed with three coins. List S, $\mathbb{F}$, and μ. You may wish to note that the sample space has eight elements, and, for example, the outcome *HTH* is different from the outcome *HHT*. Both have $Y = 2$.

2. Let W be the continuous (exponential) random variable that assigns values x in the set $[0, \infty)$ according to the density function e^{-x}. Find the cumulative probability function $P[X \leq x]$ for any $x \in [0, \infty)$.

Practice Test on Chapter 4

(These are take-home, i.e. tough, problems.)

1. A $\sigma - algebra$ over a set X is a collection of subsets of X that is closed under countable unions, countable intersections, and complementation. If we partition $[0, 1]$ into the two disjoint sets $S = [0, 1/2)$ and $T = [1/2, 1]$, how many elements are in the resulting σ-algebra generated by S and T (by taking every possible countable union, countable intersection, and complement of those sets and resulting sets)?

2. If we take every singleton set $\{x\}$ for every $x \in \mathbb{R}$, and then take every possible countable union, countable intersection, and complement of those sets and resulting sets (i.e. the algebra *generated* by the singleton sets), is this the same algebra as the σ-algebra generated by every possible interval on $\mathbb{R}$? Why or why not?

3. Define the function $\rho : [0, 1] \to \mathbb{R}^+$ by $\rho(x) = x^2$. Define the measure of subintervals of $[0, 1]$ by

$$\mu_\rho([a, b]) = \rho(b) - \rho(a) = b^2 - a^2.$$

   Is the measure μ_ρ universal? Countably additive? Translation invariant? Does it satisfy the property of length agreement?

4. Use the three-step process to find $\int_0^1 (x + 1) d\mu_\rho$.

5. Find the angle between $x + 1$ and x^2 in $L^2(\mu_\rho)$, where μ_ρ is defined as in the previous problem.

6. Toss two coins, and then have a machine (which is highly irregular and works unreliably) fill a one-gallon paint can with a volume x of paint. Suppose any value x between 0 and 1 gallon is the filled amount of paint randomly attained with equal weight. Let X be (the number of heads tossed + x). Describe the probability space for X and find a formula for the cumulative probability of X.

Chapter 5

Hilbert Space Operators

5.1 Bounded Linear Operators on L^2

Key Concepts

1. A Hilbert space *operator* is a bounded linear map from one Hilbert space to another.

2. An operator T is *linear* when, for $f, g \in \mathcal{H}$ and scalar constants a and b, $T(af + bg) = aT(f) + bT(g)$. It is *bounded* when there exists a constant $K > 0$ such that, for every f in $\mathcal{H}$, $\|T(f)\|_{\mathcal{G}} \leq K \cdot \|f\|_{\mathcal{H}}$.

3. A *rank-one operator* T on L^2 has the form $T(f)(x) = \langle f, \phi \rangle \phi$ for a given $\phi \in L^2$.

4. Two operators $T : \mathcal{H} \to \mathcal{H}$ and $S : \mathcal{G} \to \mathcal{G}$ are *isometrically isomorphic* when there is an isometry U from one of the operator's domain Hilbert space onto the other's that makes the action of the operators equivalent, meaning $\langle Tf, g \rangle_{\mathcal{H}} = \langle UTU^{-1}u, v \rangle_{\mathcal{G}} = \langle Su, v \rangle_{\mathcal{G}}$ for all $f, g \in \mathcal{H}$ and $u, v \in \mathcal{G}$ with $Uf = u$ and $Ug = v$.

Broader Descriptions Operators on a Hilbert space $\mathcal{H}$ are objects that map the elements of $\mathcal{H}$ to another Hilbert space, say $\mathcal{G}$ (where perhaps $\mathcal{H}$ and $\mathcal{G}$ are the same space)[1]. So if the operator is T, then we describe the map from one space to the other as

$$T : \mathcal{H} \to \mathcal{G}.$$

If f is an element in $\mathcal{H}$, then we write $T(f)$ as its image in $\mathcal{G}$. We will often relax the need for parentheses and write Tf for $T(f)$.

We will assume $\mathcal{H}$ is separable: it has a countable basis. This section examines the situation where $\mathcal{H}$ is an $L^2(\mu)$ space. Then an operator on $L^2(\mu)$ maps the functions in $L^2(\mu)$ to the objects in another Hilbert space $\mathcal{G}$, where perhaps $\mathcal{G} = L^2(\mu)$. In general, we say the operator T is linear when, for $f, g \in \mathcal{H}$ and scalar constants a and b,

$$T(af + bg) = aT(f) + bT(g).$$

The operator T is bounded when there exists a constant $K > 0$ such that, for every f in $\mathcal{H}$,

$$\|T(f)\|_{\mathcal{G}} \leq K \cdot \|f\|_{\mathcal{H}}.$$

The smallest such K, described in terms of the infimum over all such constants, is the norm of the operator, or the *operator norm*, and so a bounded operator is one with finite operator norm. The value of the norm for a given operator T is written $\|T\|$, and it is calculated as

$$\|T\| = \sup\{\|Tf\|_{\mathcal{G}}\},$$

[1] In the 1960s, Louis de Branges and James Rovnyak examined a certain type of functional Hilbert space, which they described as "square-summable." Their space of square-summable power series is a vector space over the complex numbers that admits an inner product. It is naturally seen to be equivalent to ℓ^2. Studying operators on square-summable power series continues to form a productive area of investigative research.

where the supremum of the $\mathcal{G}$-norm values is taken over all elements f in $\mathcal{H}$ with $\|f\|_{\mathcal{H}} \leq 1$.

Example: Choose a function $r(x)$ in an L^2 space. Then, for f in it, define the operator $T_r : L^2 \to L^2$ by

$$T_r(f) = \langle f, r \rangle \cdot \ r(x).$$

You can see that every function in L^2 is mapped to a constant multiple of $r(x)$. Hence (considering L^2 as a vector space of functions), the range of T_r (the set of output values) consists of the one-dimensional span of the function r. (The one-dimensional span of r is as defined in any linear algebra course as the set of functions g of the form $g(x) = c \cdot r(x)$ for a constant c. In this case, the output values of T_r are exactly that type, with $c = \langle f, r \rangle$.) Because its range is one-dimensional, the operator T_r is a *rank-one operator*. It turns out T_r is a bounded linear operator, and we can show why each of the two properties hold.

Linearity: T_r is linear because, for L^2 functions f and g and constants a and b,

$$T_r(af + bg) = \langle af + bg, r \rangle \cdot \ r(x) = a\langle f, r \rangle \cdot \ r(x) + b\langle g, r \rangle \cdot \ r(x) = aT_r(f) + bT_r(g).$$

Boundedness: We can see T_r is bounded by examining the L^2-norm of any $f \in L^2$:

$$\|T_r(f)\| = \|\langle f, r \rangle \cdot \ r(x)\| = |\langle f, r \rangle| \cdot \|r(x)\|.$$

By Schwarz's inequality (see p. 140), $|\langle f, r \rangle| \leq \|f\| \cdot \|r\|$, and so

$$\|T_r(f)\| \leq (\|f\| \cdot \|r\|) \cdot \|r\| = \|r\|^2 \cdot \|f\|.$$

Boundedness is therefore satisfied by setting $K = \|r\|^2$. This value is the operator norm $\|T_r\|$, since $\sup\{\|T_r(f)\|\} = \|r\|^2$ is attained when $\|f\| = 1$.

Problems. 1. For the function space $L^2(-\pi, \pi)$, define $T_{\cos x}(f) = \langle f, \cos x \rangle \cdot \cos x$. What is $T_{\cos x}(f)$ when $f(x) = x^2$? Determine the operator norm $\|T_{\cos x}\|$ and show $\|T_{\cos x}(x^2)\| \leq \|T_{\cos x}\| \cdot \|x^2\|$.

2. The shift operator S on ℓ^2 is defined for a vector $\vec{v} \in \ell^2$ by $S(\vec{v}) = S(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{bmatrix}) = \begin{bmatrix} 0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix}$. That is, it shifts each component of the vector down one spot and places a 0 in the first component. We take up the shift operator for an arbitrary Hilbert space in Section 5.3. Determine $S(\vec{v})$, where $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$.

3. Determine $S(\vec{v})$, where $\vec{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$, where the 1 appears in the nth coordinate spot.

5.2 Bounded Hilbert Space Operators

Key Concepts

1. The form of a *linear operator on a finite-dimensional Hilbert space* can always be described as matrix multiplication.

2. A *rank-one operator* T on a given Hilbert space $\mathcal{H}$ maps a general element $u \in \mathcal{H}$ to $Tu = \langle u, v \rangle v$, where v is a chosen element of $\mathcal{H}$.

3. An operator $T : \mathcal{H} \to \mathcal{G}$ is *compact* when it can be written as $T(f) = \sum_{n=0}^{\infty} \lambda_n \langle g_n, f \rangle \phi_n$ for $f \in \mathcal{H}$, where the vectors $g_0, g_1, g_2, \ldots$ and $\phi_0, \phi_1, \phi_2, \ldots$ are (not necessarily complete) orthonormal sets. Here $\lambda_0, \lambda_1, \lambda_2, \ldots$ forms a sequence of nonnegative values—the eigenvalues of T. An excellent example is $T(f)(x) = \int_0^{\pi} \sin(x+y) f(y)\, dy$ on $L^2(0, \pi)$.

Four important examples of infinite-dimensional Hilbert spaces

1. For real-valued elements $u, v \in L^2(\mathbb{R})$, $\langle u, v \rangle = \int_{\mathbb{R}} uv$.

2. For ℓ^2, $\langle \vec{u}, \vec{v} \rangle = \sum_{j=0}^{\infty} u_j v_j$, where u_j and v_j are the jth (real-valued) entries in the corresponding vector.

3. For complex-valued analytic functions $f, g \in H^2(\mathbb{D})$ that have unit circle boundary functions $f(\mathrm{e}^{it})$ and $g(\mathrm{e}^{it})$, respectively, $\langle f, g \rangle_{H^2(\mathbb{D})} = \frac{1}{2\pi} \int_0^{2\pi} f(\mathrm{e}^{it}) \overline{g(\mathrm{e}^{it})}\, dt = \sum_{n=0}^{\infty} a_n \overline{b_n}$, where $f(\mathrm{e}^{it}) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{int}$ and $g(\mathrm{e}^{it}) = \sum_{n=0}^{\infty} b_n \mathrm{e}^{int}$.

4. For real-valued elements u, v in the Sobolev space $W^{1,2}$, $\langle u, v \rangle_{W^{1,2}} = \int_a^b u(x) \cdot v(x)\, dx + \int_a^b Du(x) \cdot Dv(x)\, dx$.

The Jordan Form Theorem If an $n \times n$ matrix T has m linearly independent eigenvectors, then there exists an $n \times n$ invertible matrix P such that $PJP^{-1} = T$, where J is a block diagonal matrix of the form $J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix}$. (Any blank part of the matrix is assumed to be filled with entries that are all zero.) J_i corresponds to the ith linearly independent eigenvector and is a block matrix of the form $J_i = \begin{bmatrix} \lambda_i & 1 & \\ & \ddots & \vdots & 1 \\ & & \lambda_i \end{bmatrix}$, where λ_i is the ith eigenvalue (and the blank components are assumed to be filled with zeros).

Examples. 1. Suppose $T = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 5 & -3 & 1 \end{bmatrix}$. It has two eigenvalues $\lambda = 4$ (which has multiplicity two—in this case, it appears twice on the diagonal and T is lower-triangular) and $\lambda = 1$. Find two eigenvectors for T, one for each eigenvalue.
Solution: Set $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 4 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and equate coefficients to solve for x, y and z, using an arbitrary value, say 1 or -1, for any free variable. Repeat with $\lambda = 1$. For example, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are two solutions.

2. What is the Jordan form matrix J?
Solution: Because these are the eigenvectors in the last problem are the only two up to arbitrary constant multiples, the other column of P must be a generalized eigenvector as described in the footnote on p. 228. The next exercise shows it is a generalized eigenvector that goes with $\lambda = 4$. Then the Jordan block for $\lambda = 4$ is 2×2, and $J = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise Suppose $T = \begin{bmatrix} 4 & -1 & 0 \\ .4 & 3.4 & .2 \\ .2 & -5.8 & 1.6 \end{bmatrix}$. It has only one eigenvalue $\lambda = 3$ (which has multiplicity three as a zero of the characteristic polynomial $p(x) = -(x-3)^3$).

1. Show $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$ is an eigenvector for T, in that it solves the eigenvector equation $T\vec{v} = \lambda \vec{v}$.

2. Given $\vec{v}$ is the only eigenvector for T, what is the Jordan form matrix J?

3. Using $P = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ -4 & -1 & 1 \end{bmatrix}$, which has inverse $P^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -4 & -2 \\ -1 & 9 & 2 \\ 3 & -7 & -1 \end{bmatrix}$, prove T satisfies $T = PJP^{-1}$ for the matrix J you constructed in part 2.

5.3 The Unilateral Shift Operator

Key Concepts

1. The *shift operator* M_z on $H^2(\mathbb{D})$ acts according to $M_z(f)(z) = z \cdot f(z)$. The boundary function for $M_z(f)(z)$ is $\mathrm{e}^{it} f(\mathrm{e}^{it})$. Hence M_z induces a shift operator S on $H^2(\mathbb{T})$ defined as $S(f)(\mathrm{e}^{it}) = \mathrm{e}^{it} f(\mathrm{e}^{it})$.

2. A bounded linear operator S on a Hilbert space $\mathcal{H}$ is a *unilateral shift operator* if S is an isometry and $\lim\limits_{n\to\infty} \|(S^*)^n f\|_{\mathcal{H}} = 0$ for all $f \in \mathcal{H}$.

3. An *outer function* on $H^2(\mathbb{D})$ is of the form $g(z) = c\,\mathrm{e}^{\frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{e}^{it}+z}{\mathrm{e}^{it}-z} \log|g(\mathrm{e}^{it})|\,\mathrm{d}t}$

4. An *inner function* on $H^2(\mathbb{D})$ is either (a) a so-called Blaschke function, such as $B(z) = \prod\limits_n \frac{a_n - z}{1 - a_n z}$, where $a_n = 1 - 2^{-n}$ (and a second, simple example is $B(z) = z$), or (b) a so-called singular inner function, such as $s(z) = \mathrm{e}^{\frac{z+1}{z-1}}$. These functions have norm one on the boundary of the disk.

5. An invariant subspace $\mathcal{M}$ is *invariant* for a given operator A for a given operator A when $Zx \in \mathcal{M}$ for any $x \in \mathcal{M}$.

Inquiry. For a given operator A on a Hilbert space $\mathcal{H}$, what are the invariant subspaces for A?[2]

A partial answer comes from Buerling's Theorem.

Beurling's Theorem: The (nontrivial) invariant subspaces of M_z (the operator multiplication by z) on $H^2(\mathbb{D})$ are characterized as $bH^2 = \{b(z) \cdot f(z) : \ f \in H^2(\mathbb{D})\}$, where b is an inner function in $H^2(\mathbb{D})$.

Inquiry. Does every (bounded linear) operator A on a Hilbert space $\mathcal{H}$ have a nontrivial invariant subspace?

Answer: No one knows. This is the most famous open problem in function theory. Though it has been worked on by mathematicians in earnest for well more than 75 years, no one has been able to figure it out.

Example Let b be an inner function on $H^2(\mathbb{D})$. $M = b(z)H^2(\mathbb{D})$ for $S = M_{z^2}$ on $H^2(\mathbb{D})$, defined by $S(f)(z) = z^2 \cdot f(z)$. Then M is invariant for S because, for any $x \in M$, $x = b(z) \cdot f(z)$ for some $f \in H^2(\mathbb{D})$. Then $Sx = z^2 b(z) \cdot f(z) = b(z) \cdot z^2 f(z)$. But $z^2 f(z)$ is also in $H^2(\mathbb{D})$ (in fact, $\|z^2 f(z)\|_{H^2(\mathbb{D})} = \|f(z)\|_{H^2(\mathbb{D})}$), and so $Sx \in b(z)H^2(\mathbb{D}) = M$.

Exercises. Say why the subspace M is invariant for the indicated shift operator S.

1. $M = \mathrm{e}^{it} H^2(\mathbb{T})$ for the shift operator $S(f)(\mathrm{e}^{it}) = \mathrm{e}^{it} f(\mathrm{e}^{it})$, where $f \in H^2(\mathbb{T})$.

2. $M = \left\{ \begin{bmatrix} b(z)g \\ 0 \end{bmatrix} : g \in H^2(\mathbb{D}) \right\}$, where b is an inner function in $H^2(\mathbb{D})$, for the shift operator S defined in Exercise 8.

[2] The nontrivial ones, of course – $\emptyset$ and $\mathcal{H}$ are always invariant

5.4 The Spectral Theorem

Key Concepts

1. A *self-adjoint* operator T (on a Hilbert space $\mathcal{H}$) – an important class of operators with many special properties

2. A *multiplicity one* operator

3. A *Hilbert space isometry* that gives an equivalence between $\mathcal{H}$ and $L^2(\mu)$ for some Borel measure μ, where the action of T is as simple as possible

4. The *Spectral Theorem* that describes the interaction between these objects

Inquiry. Can we "diagonalize" a self-adjoint operator, even one that acts on an infinite-dimensional Hilbert space? What would such a diagonalization look like?

The Spectral Theorem: If T is a self-adjoint bounded linear operator on a Hilbert space $\mathcal{H}$ with multiplicity one, then there exists a Borel measure μ on $\mathbb{R}$ and a Hilbert space isometry $U : \mathcal{H} \to L^2(\mu)$ such that $UTU^{-1}f(x) = xf(x)$ for $f \in L^2(\mu)$.

Examples: 1. Suppose T acts as self-adjoint (it is its own conjugate-transpose) matrix mulitiplication acting on the finite-dimensional vector space (say, $\mathbb{R}^n$). If, for example, the matrix has n distinct eigenvalues λ_k with corresponding eigenvectors $\vec{e}_k$, then we can construct μ as the sum of n point masses at the λ_k's. The isometry U is constructed from the eigenvectors, and the Spectral Theorem is a special case of the Jordan Form Theorem in Section 5.2.

2. Suppose T is a multiplicity one Toeplitz operator acting on ℓ^2 that is self-adjoint (again, it is its own conjugate-transpose). (Equivalently, $T = T_w$ is multiplication by the real-valued function w on $H^2(\mathbb{D})$ followed by projection onto $H^2(\mathbb{D})$.) Can we know what the Spectral Theorem's constructions are? The answer was impressively described in the 1960s by American mathematician Marvin Rosenblum. Here is his result:

1. The spectral measure μ is absolutely continuous, so $\mu(x) = m(x)\,dx$ for some (almost everywhere) continuous function $m(x)$ on $\mathbb{R}$. Hence, a description of the spectral measure μ is determined from a description of its weight function $m(x)$.

2. The support of the spectral measure μ (the portion of the real line where $m(t) \neq 0$) is the spectrum of T and is denoted $sp(T)$. It is equal to the interval $[c, d]$, where $c = \min w(\mathrm{e}^{it})$ and $d = \max w(\mathrm{e}^{it})$.

3. Write the set $E_x = \{\mathrm{e}^{it} : w(\mathrm{e}^{it}) \geq x\} = \{\mathrm{e}^{it} : a(x) \leq t \leq b(x)\}$ for some real functions $a(x)$ and $b(x)$. Then
$$m(x) = \pi^{-1} \sin\left(\frac{b(x) - a(x)}{2}\right) \quad \text{on} \quad [c, d].$$

4. The isometry U, defined on kernels $\vec{k_z} \in \ell^2$ and for almost all real x according to
$$U(\vec{k_z})(x) = [\overline{g(z,x)}\sqrt{1 - \overline{z}\mathrm{e}^{ia(x)}}\sqrt{1 - \overline{z}\mathrm{e}^{ib(x)}}]^{-1},$$
sets up T_w as unitarily equivalent (via the Hilbert space isomorphism U) to multiplication by x on $L^2(\mu)$. That is, $UTU^{-1}f(x) = xf(x)$ for $f \in L^2(\mu)$. Here, $|w(\mathrm{e}^{it}) - x| = |g(\mathrm{e}^{it}, x)|^2$, where g is outer and $x \in \mathbb{R}$ satisfies $\frac{1}{2\pi}\int_0^{2\pi} |\log|w(\mathrm{e}^{it}) - x\|\,dt < \infty$. Furthermore, $U^{-1}(f)(z)$ is the vector in ℓ^2 formed from the Taylor series coefficients of $h(z) = \int_c^d f(x)\overline{U(\vec{k_z})(x)}\,d\mu(x)$.

Sample Test Problems for Chapter 5

1. Define the operator $Sf = \begin{bmatrix} z^2 f_1(z) \\ z f_2(z) \end{bmatrix}$ on $\mathcal{H}$, where $f(z) = \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} \in \mathcal{H} = H^2(\mathbb{D}) \times H^2(\mathbb{D})$. Show S is a unilateral shift operator.

2. Let T be the so-called "Volterra operator" on $L^2(0,1)$ defined as $T(f)(x) = \int\limits_0^x f(y)\,dy$.

   (a) Prove T is linear and bounded.

   (b) What is $T(x^2)$? $T(\cos x)$?

3. Show, if $c \in (0,1)$, the subspace $A = \{f :\ f(x) = 0$ for almost all $x \in (0,c)\}$ is invariant for the Volterra operator defined in the last exercise.

4. The multiplication operator T on $L^2(0,1)$ is $T(f) = r(x) \cdot f(x)$ for a given $r \in L^2(0,1)$.

   (a) Prove T is linear.

   (b) Suppose $r(x)$ is continuous on the interval $[0,1]$, so it attains its maximum. Define M so $|r(x)| \leq M$ for $x \in [0,1]$. Prove $\|T(f)\|_{L^2(0,1)} \leq M \cdot \|f\|_{L^2(0,1)}$.

   (c) Now additionally assume $M = |r(x_0)|$ for some $x_0 \in [0,1]$. Define a sequence of functions $g_n(x) = \sqrt{n/2}$, if $x \in [x_0 - 1/n, x_0 + 1/n]$ (and 0 otherwise), $n = 1, 2, 3, \ldots$. Find the $L^2(0,1)$-norm of g_n.

   (d) Find an expression (in terms of the integral) for $\|T(g_n)\|_{L^2(0,1)}$, where g_n is as defined in part (c).

   (e) Use the fact that, for f continuous at x_0, $(n/2)\int_{x_0-1/n}^{x_0+1/n} f(t)\,dt \to f(x_0)$ to evaluate your expression in part (d) to find $\lim\limits_{n\to\infty} \|T(g_n)\|_{L^2(0,1)}$.

   (f) Use your results of parts (b) and (e) to show $\|T\| = \max\{|r(x)| :\ 0 \leq x \leq 1\}$.

5. Give any example of a compact bounded linear operator on the Hardy space $H^2(\mathbb{D})$. Make sure you say why the operator is compact, and prove your operator is linear and bounded.

Sample Test Problems for Final Exam

1. Use the standard construction to find $\int_{-2}^{1} x^2\, dx$.

2. Use the Riemann-Lebesgue theorem to determine if the following Riemann integral exists.

$$R\text{-}\int_0^1 f(x)\, dx, \text{ where } f(x) = \begin{cases} 1 & \text{if } x \text{ is in the Cantor set } C \\ 0 & \text{otherwise} \end{cases}$$

   Say why or why not, including identification of the function's points of discontinuity. Also, if it exists, find its value.

3. Find each integral $\int f$ by applying either the Monotone Convergence Theorem or the Lebesgue Dominated Convergence Theorem. As you use either theorem for each integral, make sure you explicitly state why each of the theorem's assumptions holds.

   a) $\int_0^1 \frac{1}{\sqrt{x}}\, dx$

   b) $\int_0^\infty x\mathrm{e}^{-x^2}\, dx$

4. Use the Monotone Convergence Theorem to prove the following:

   For $f \in L^1$ with $f \geq 0$, if $\int f = 0$, then $f = 0$ almost everywhere.

5. A. Each of the following functions are measurable. Find their L^p-norms in the given function space.

   i $f(x) = \cos x$ in $L^1(-\pi, \pi)$

   ii $m(x) = 1/x^4$, $x \geq 1$ in $L^2(\mathbb{R})$

   B. Find the L^2-distance between $s(x) = 2\sqrt{x}$ and $t(x) = 1 + x^2$ in $L^2(0, 1)$.

6. Let X, a random variables with Borel distribution, be the number of Aces dealt in a well-shuffled two-card deal. Using combinatoric formulas, it turns out there are 1326 hands. The probability of dealing two Aces is $1/221$, one Ace is $32/221$, and no Aces is $188/221$.

7. a Describe (don't give a complete list) the elements of the sample space S, and explicitly write out one of them as a representative example.

   b Identify (don't give a complete list) the elements of the σ-algebra of subsets $\mathbb{F}$, and explicitly write out **three** of them as a representative example.

   c Describe in detail the probability measure μ.

   d For your measure μ in Part (c), determine $\int x^2 \, d\mu$.

8. Let $k(x, y)$ be real-valued and continuous for x and y in $(0, 1)$, so $\int_0^1 \int_0^1 |k(x, y)|^2 \, dx \, dy = M < \infty$ for some constant M. For $f \in L^2(0, 1)$ define the operator T by

$$T(f)(x) = \int_0^1 k(x, y) \cdot f(y) \, dy.$$

   a Show T is linear.

   b Use the Schwarz inequality ($|\langle f, g\rangle| \leq \|f\| \cdot \|g\|$) to prove T is bounded.

   c When $k(x, y) = xy$ and $f(x) = x^2$, evaluate $T(f)(x)$. In other words, find $T(x^2)$.

9. Prove the **Riesz-Fischer Theorem:** If $\{f_n\}$ is a Cauchy sequence of functions in an L^p space with $p \geq 1$ (so that, given $\varepsilon > 0$, there exists N so that $\|f_n - f_m\|_p < \varepsilon$ whenever $m, n \geq N$), then f_n converges to a function $f \in L^p$ in the L^p-norm limit.
In your proof, feel free to use Lemma 3.1.2: Let $f_k \in L^p$ for $k = 1, 2, 3, \ldots$. If $\sum \|f_k\|$ converges as a real-valued series, then $\sum f_k$ converges to a function g in L^p. Moreover, $\sum f_k(x)$ converges to $g(x)$ pointwise for almost all x.

Chapter 6

A Few Ideas for Research Projects

A student who has finished a course on the Lebesgue integral may naturally be interested in pursuing an undergraduate collaborative research project. The possibilities are endless, but here are just a few ideas that follow immediately from concepts presented in such a course. Many are not developed in a specific manner, but they are offered with a simple hope that they might be helpful, and that they might encourage an increase in the number of such undergraduate investigations.

1. A student can look at standard operators, such as a composition operator, on various Hilbert spaces. A wonderful undergraduat project would be to study operators on $L^2(\mu)$ where μ is a point mass measure, with either a countable number of point masses (fairly difficult in many cases) or a finite number (which could generate some interesting results. These projects would especially builds off of material presented in Section 4.2 of this Resource Guide.

2. A student could nicely determine orthogonal basis structures for various $L^2(\mu)$ spaces. Section 3.3 and 3.4 look at such bases for Lebesgue measure, specifically for $L^2(\mathbb{R})$ and $L^2(0,1)$. Those could serve as models for students to explore other L^2 spaces, including spaces where the measure is not absolutely continuous.

3. A student could get curious about various probability spaces that would correspond to real-life scenarios. Many random variables are still not understood well mathematically. In 2015-16, Bill Johnston mentored Alex Olivero at Butler University on a project that looked at issues connected with the average payoff random variable on repeated plays of the St. Petersburg Paradox. Alex presented an award-winning poster at the JMM on his work.

4. Hilbert spaces of course include finite-dimensional ones, and so the topic of operators on Hilbert spaces include matrix multiplication operators on finite-dimensions. The material in Chapter 5 launches investigations on this material. Many projects are possible for interesting undergraduate investigations. For example, and in brief:

   (a) What is the Jordan form of various categories of matrices?

   (b) What properties are involved for matrices that are isometries or partial isometries?

   (c) What are the invariant subspaces for interesting examples of matrix multiplication?

5. Operators on infinite dimensional Hilbert spaces are challenging, but questions can be tractable. A student might study a restricted category of operators, such as a restricted category of compact operators, and see what generalizations can be made about them.

Complicated operators can be made simpler by projecting them down onto finite-dimensional subspaces of the Hilbert space, and then linear algebraic techniques can be applied.

6. Describing relationships between Hilbert spaces can be interesting for students. Of course, finite-dimensional Hilbert spaces are isomorphic to each other when their dimensions are equal. But finding a formula for the Hilbert space isomorphism that takes one onto the other, and the inverse isomorphism, can be a great investigation for students when the vectors in the spaces are quite different objects such as column vectors and functions.

7. Structures of singular continuous measures and associated spaces are incredibly active areas of research. (For example, see [2].) A student could formulate a singular continuous measure μ of her/his own that is different from the Cantor set, and then ask questions about it, such as its value for $\int x \, d\mu$.

8. Nonmeasurable sets can be an area of interest. Their constructions can be varied. A student may wish to construct an example different from the Vitali set of a nonmeasurable set.

9. The Cantor set is clearly only one example of a measure zero set of uncountable cardinality. Relationships between set cardinality and measure zero (either for Lebesgue measure or some other Borel measure) could provide fruitful investigations. For starters, a student might wish to construct a number of measure zero uncountable real-valued sets, thinking about algorithmic constructions that might provide assistance with these constructions. Then their non-Lebesgue measure could be found, using other Borel measures μ.

10. There can be new formulas or insights in working with kernel functions on various Hilbert spaces. Almost any undergraduate professor has in the past assumed topics such as de Branges-Rovnyak spaces are far out of reach of undergraduate students, and in fact these types of topics may be new to almost any undergraduate faculty mentor. But they are very accessible for a student who has worked through introductory material on operators, and they should be accessible to both mentor and student for some interesting work. Indeed, de Branges-Rovnyak spaces form a topic that is really hot right now – many items in operator theory and function theory seem to be explained well by them. An excellent survey, which an undergraduate student can now understand, is at [1].

11. Here's a space to fill in your own interests and ideas for student collaborative research work:

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[4] Weir, Alan J., *Lebesgue Integration and Measure*, Cambridge University Press, Cambridge, 1996. ISBN: 0-521-09751-7