Geometric Limits of Julia Sets of Iterates and Powers of Polynomials

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Definitions

For background and further reading, see [5, 1]. For the purposes of this poster,

- The **unit disk**,
$$\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$$
is the set of all points within 1 unit, exclusive, of the origin;
- The **orbit** of a point z
$$\left\{f^k(z)\right\}_{k=0}^{\infty}$$
is the sequence of successive maps of z under f;
- The **filled Julia set** for a given polynomial map $f\colon \mathbb{C} \mapsto \mathbb{C}$,
$$K(f) = \left\{z \in \mathbb{C} \colon \{f^k(z)\}_{k=0}^{\infty} \text{ is bounded}\right\}$$
is the set of points whose orbits by f remain bounded;
- The **Julia set** for a map f,
$$J(f) = \partial K(f)$$
is the boundary of the filled Julia set;
- The **Fatou set** for a map f,
$$\mathscr{F}(f) = \mathbb{C} \backslash J(f)$$
is the compliment of the Julia set;
- The **quibble** of a polynomial p,
$$\mathscr{Q} = p^{-1}(\mathbb{D})$$
is the preimage of the disk under p;
- The **extended complex plane**,
$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$
is the complex plane together with the point at infinity;
- A **periodic point** $z \in \hat{\mathbb{C}}$ is periodic for f with period k if $f^k(z) = z$ and the points $z, f(z), \ldots, f^{k-1}(z)$ are all distinct;
- The **multiplier** λ **of a periodic point** z of period k,
$$\lambda = (f^k)'(z) = \prod_{j=0}^{k-1} f'(f^j(z))$$
where if $|\lambda| < 1$, then z is attracting; if $|\lambda| > 1$, z is repelling; if $|\lambda| = 1$, z_0 is indifferent;
- A point $z \in \hat{\mathbb{C}}$ is **preperiodic** if, for some $i > j > 0$,
$$f^i(z) = f^j(z);$$
- An **attracting basin** is a component $A_j \subseteq \mathscr{F}(f)$ where $f^p(A_j) = A_j$ with some attracting periodic point $w \in A_j$ such that $f^{np}(A_j) \to w$ as $n \to \infty$; and
- A polynomial is **hyperbolic** if every component of the Fatou set is an attracting basin. Thus, $K(f) = \bigcup_{j=1}^{k} A_j$.
- For hyperbolic p, orbits of all points in $\mathscr{F}(p)$ will be attracted to a periodic attracting cycle, which we denote $z_1, \ldots, z_k$.

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Abstract

For maps of one complex variable, $f_n(z) = (p(z))^n + q(z)$, where p and q are polynomials, we show that the geometric limit as $n \to \infty$ of the filled Julia set of f_n is almost always defined and determined by preimages of the unit disk by p. Additionally, for maps g_n given as the sum of two polynomials of which one is composed with itself n times, we show that the geometric limit as $n \to \infty$ of the filled Julia set almost always fails to exist.

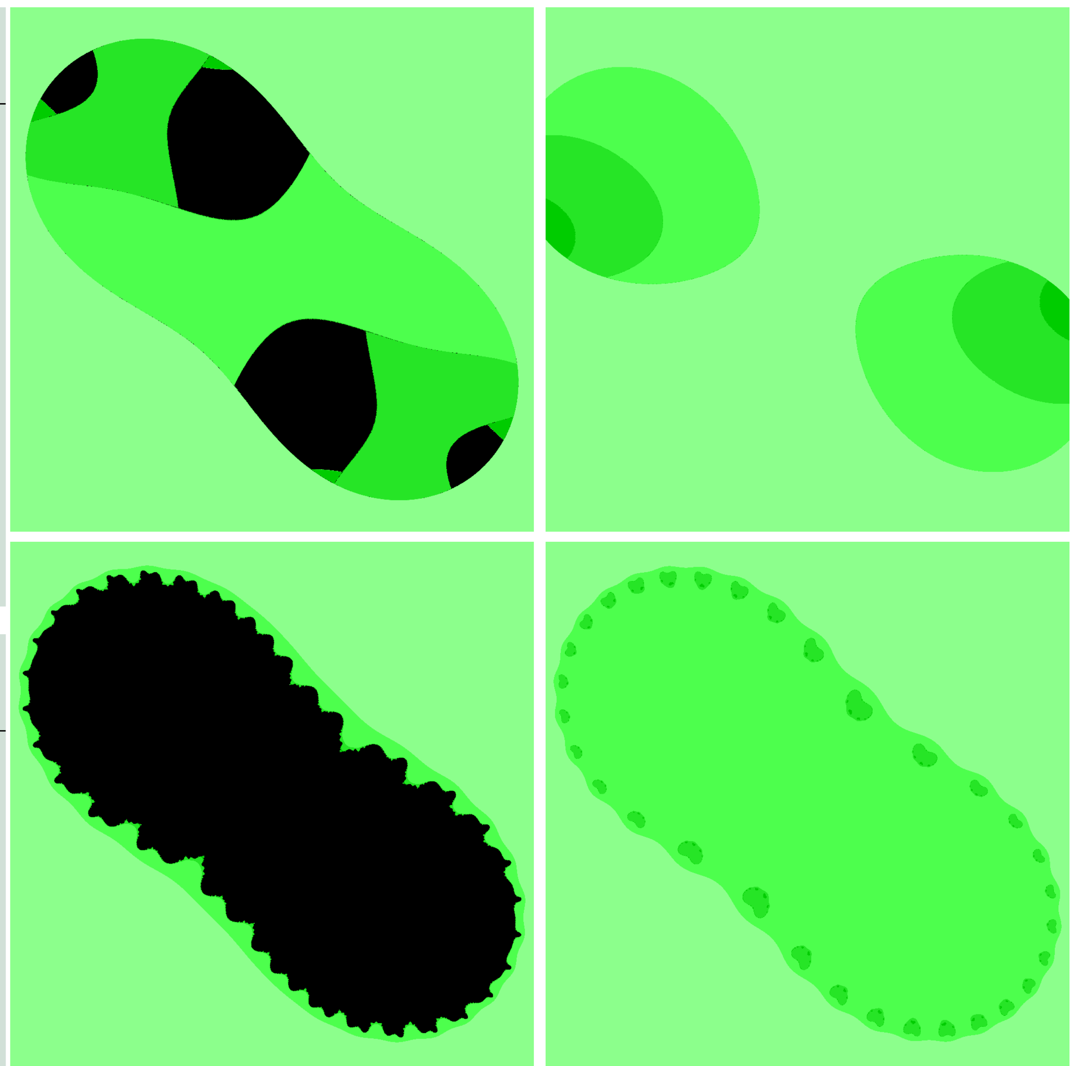
Sums with Powers: $f_n(z) = (p(z))^n + q(z)$

Theorem

Let $f_n(z) = (p(z))^n + q(z)$, where p, q are polynomials with $\deg p \geq 2$, and q hyperbolic with no attracting periodic points on $\partial p^{-1}(\overline{\mathbb{D}})$ if $\deg q \geq 2$. If $p(q(p^{-1}(\mathbb{D}))) \cap \partial\mathbb{D} \neq \emptyset$, then
$$\lim_{n\to\infty} K(f_n) = \bigcap_{j=0}^{\infty} q^{-j}(p^{-1}(\mathbb{D})) \cup \bigcup_{j=0}^{\infty} \mathscr{Q}_j$$
where
$$\mathscr{Q}_j = \left\{z \colon q^k(z) \in p^{-1}(\mathbb{D}) \text{ for } k < j-1, \text{ and } q^j(z) \in \partial p^{-1}(\mathbb{D})\right\}$$

Corollary

1. If $p(q(p^{-1}(\mathbb{D}))) \subset \mathbb{D}$, then
$$\lim_{n\to\infty} K(f_n) = p^{-1}(\mathbb{D})$$
2. If $p(q(p^{-1}(\mathbb{D}))) \subset \mathbb{C} \backslash \mathbb{D}$, then
$$\lim_{n\to\infty} K(f_n) = \partial p^{-1}(\mathbb{D})$$



Right, top: $K(f_{3200})$ for two distinct families f_n.
Right, bottom: $K(f_{16})$ for f_n where $p(q(p^{-1}(\mathbb{D}))) \subset \mathbb{D}$ (left) and $p(q(p^{-1}(\mathbb{D}))) \subset \mathbb{C}\backslash\mathbb{D}$ (right).

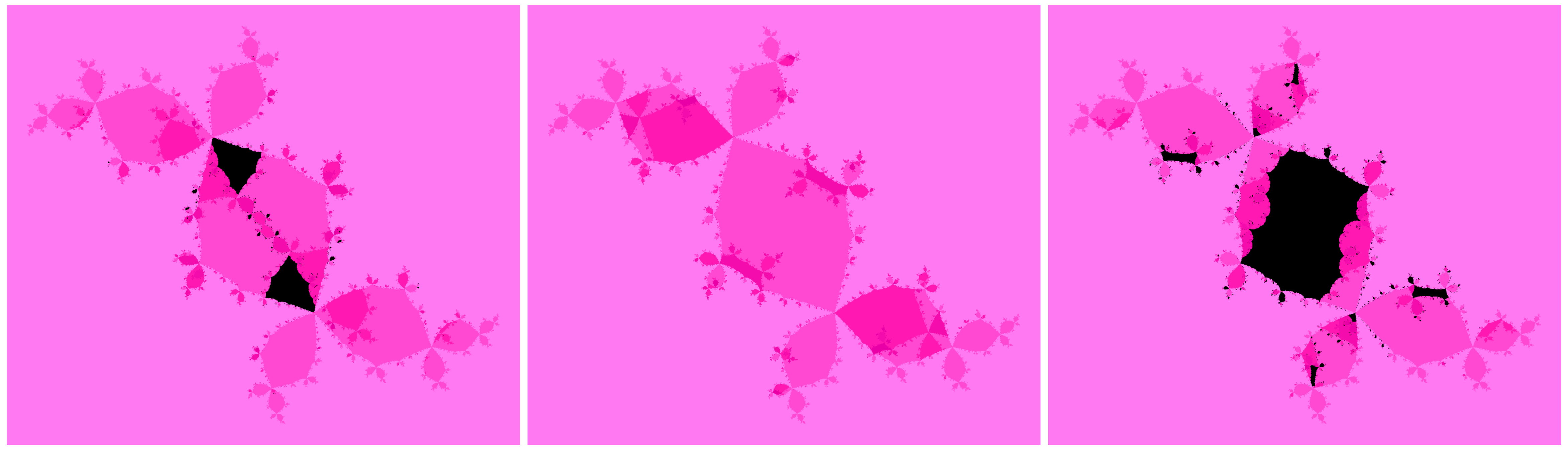
Sums with Iterates: $g_n(z) = p^n(z) + q(z)$

- Suppose p is hyperbolic with periodic attracting cycle
$$z_1, \ldots, z_k$$
- For each n, there exists some $\ell \in \{1, \ldots, k\}$ such that
$$g_n(z) \approx q(z) + z_\ell$$
- Let A_ℓ be the basin of attraction for z_ℓ of p^k, and
$$A = \bigcup_{\ell=1}^{k} A_\ell$$

Define $\hat{g}\colon A \to \mathbb{C}$ by $\hat{g}(z) = \begin{cases} q(z) + z_1, & \text{for } z \in A_1 \\ \vdots & \\ q(z) + z_k, & \text{for } z \in A_k. \end{cases}$

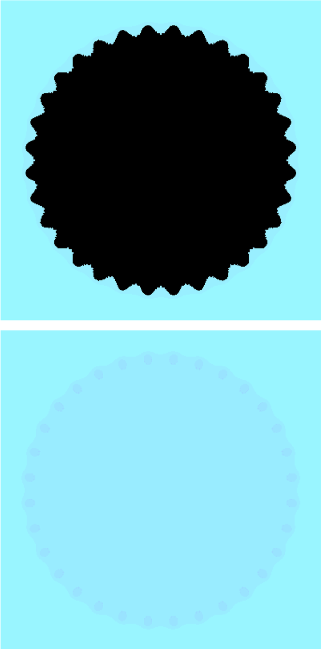
Conjecture

Suppose p is hyperbolic and $q(z) + z_\ell$ is hyperbolic for each $\ell \in \{1, \ldots, k\}$. For fixed ℓ, define the subsequence $n_m = \ell + mk$. Then
$$\lim_{m\to\infty} K(g_{n_m}) = \bigcap_{j=0}^{\infty} \hat{g}^{-j}(p^{\ell}(A)) \cup \bigcup_{j=0}^{\infty} \mathscr{J}_j,$$
where
$$\mathscr{J}_j = \{z \colon \hat{g}^j(z) \in J(p) \text{ and } \hat{g}^{\kappa}(z) \in A \text{ for } \kappa = 1, \ldots, j-1\}.$$



Above: $K(g_n)$ for $n = 49, 50, 51$ with $p(z) = z^2 - 0.1 + 0.75i$, $q(z) = z^2 - 0.2 + 0.3i$. Although the limit does not exist through $\mathbb{N}$, subsequential limits exist along classes $n \mod 3$.

An Abridged History



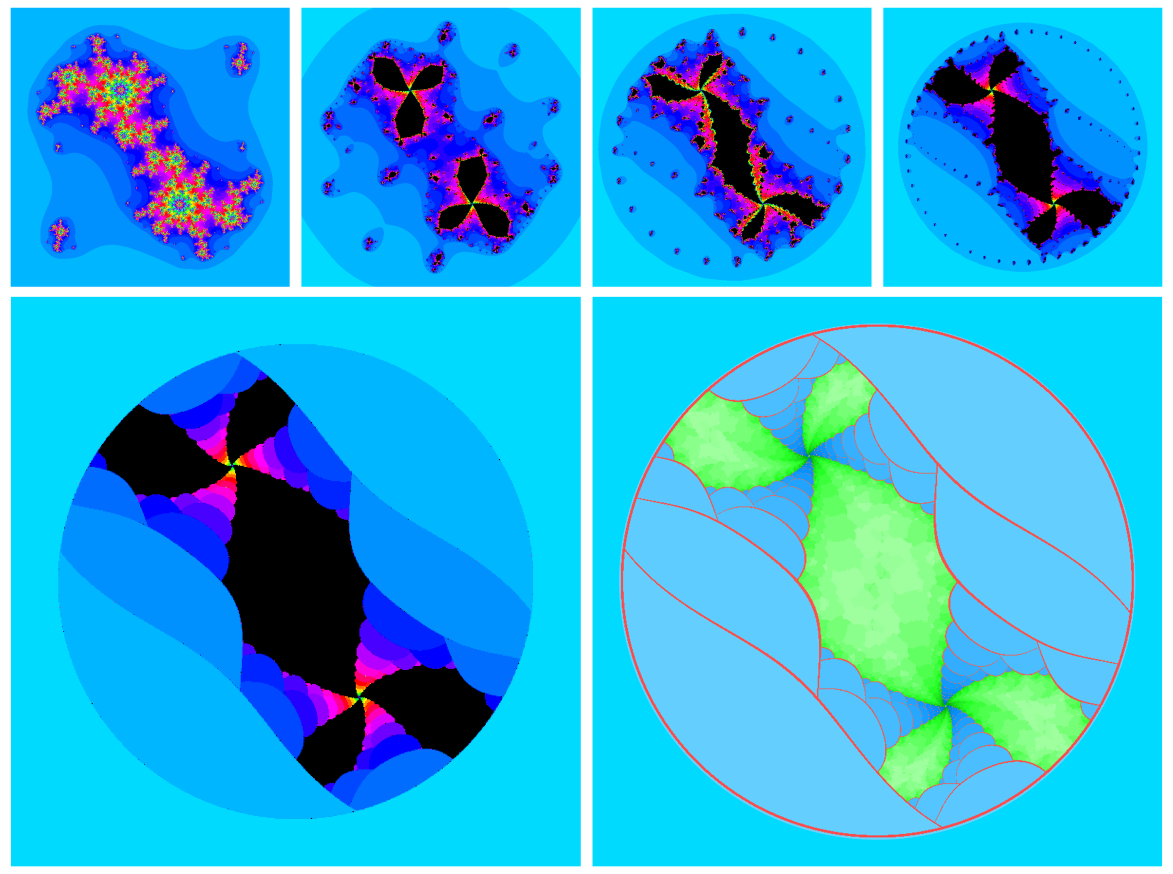
Boyd & Schulz, 2012 [2]
Let $f_n(z) = z^n + c$.

If $|c| < 1$, then $\lim_{n\to\infty} K(f_n) = \overline{\mathbb{D}}$, and

If $|c| > 1$, then $\lim_{n\to\infty} K(f_n) = \partial\overline{\mathbb{D}}$.

Kaschner, Romero, & Simmons, 2015 [4]
Let $f_n(z) = z^n + c$. For almost all $|c| = 1$, $\lim_{n\to\infty} K(f_n)$ does not exist. If $\arg c \in \mathbb{Q}$, then
$$\limsup_{n\to\infty} K(f_n) = \overline{\mathbb{D}}, \text{ and}$$
$$\liminf_{n\to\infty} K(f_n) = \partial\overline{\mathbb{D}}.$$

Brame & Kaschner, 2020 [3]
Upgrade: $g_n(z) = z^n + q(z)$. If q is hyperbolic with no attracting periodic points on $\partial\overline{\mathbb{D}}$, then
$$\lim_{n\to\infty} K(g_n) = \bigcap_{j=0}^{\infty} q^{-j}(\overline{\mathbb{D}}) \cup \bigcup_{j=0}^{\infty} S_j,$$
where $S_j = \left\{z \in \mathbb{D} \colon q^k(z) \in \mathbb{D} \text{ for } k < j-1, \text{ and } q^j(z) \in \partial\mathbb{D}\right\}$.



Top, left to right: $K(g_n)$ with $q(z) = z^2 - 0.1 + 0.75i$ and $n = 6, 12, 25, 50$. Bottom left: $K(g_{1800})$. Bottom right: Sketch of K_∞, where K_q is green and the sets S_j are red.

References

[1] Alan F. Beardon. *Iteration of rational functions*, volume 132 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Complex analytic dynamical systems.

[2] Suzanne Hruska Boyd and Michael J. Schulz. Geometric limits of Mandelbrot and Julia sets under degree growth. *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, 22(12):1250301, 21, 2012.

[3] Micah Brame and Scott Kaschner. Geometric limits of Julia sets for sums of power maps and polynomials. *Complex Analysis and its Synergies*, 6:1–8, 2020.

[4] Scott R Kaschner, Reaper Romero, and David Simmons. Geometric limits of Julia sets of maps $z^n + \exp(2\pi i\theta)$ as $n \to \infty$. *International Journal of Bifurcation and Chaos*, 25(08):1530021, 2015.

[5] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.