Geometric Limits of Julia Sets

Eleanor Waiss *(she/her)*
Butler University

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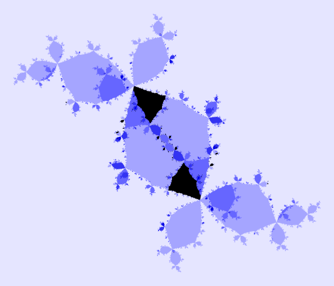

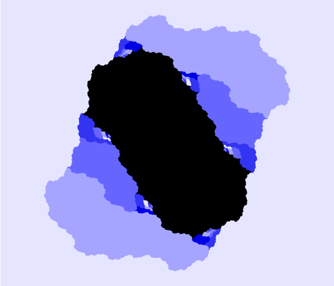

Background

For $f\colon X \to X$, we can iterate f:

$$f^n = f \circ \overset{(n)}{\cdots} \circ f$$

and consider sequences of iterates called orbits:

$$\{z_i\}_{i=0}^{\infty} = \{f^i(z_0)\}_{i=0}^{\infty} = \{z_0, f(z_0), f^2(z_0), \dots\}.$$

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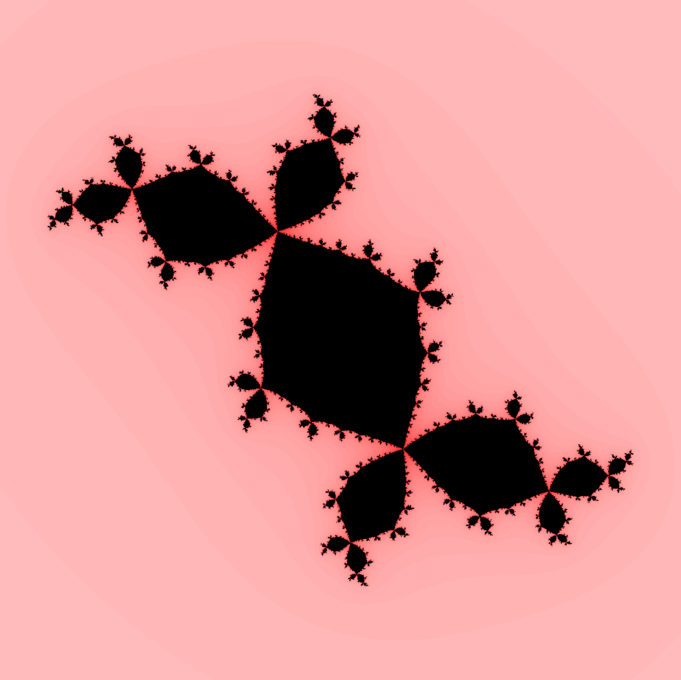
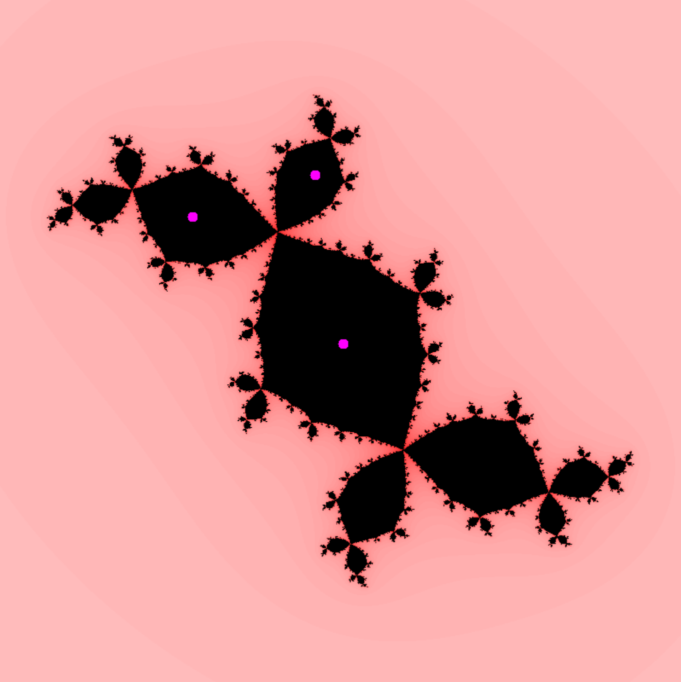
$$\{z_i\}_{i=0}^{\infty} = \{f^i(z_0)\}_{i=0}^{\infty} = \{z_0, f(z_0), f^2(z_0), \ldots\}.$$

Definition

Suppose $f: \mathbb{C} \to \mathbb{C}$ is a polynomial map. The filled Julia set, $K(f)$, is the set of points whose orbits by f are bounded.

For further reading, see [6, 1].





A brief thread through history

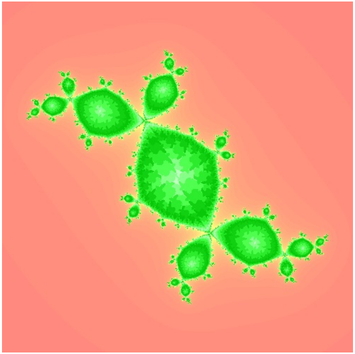
2012 [3] Boyd & Schulz: $f_n(z) = z^n + c$.

Geometric limits of Julia sets

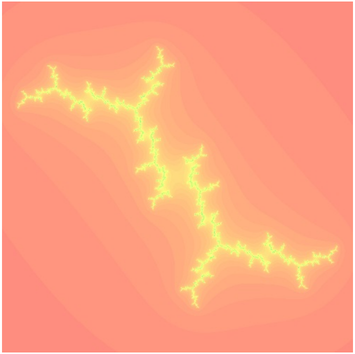
Let $f_n \colon \mathbb{C} \to \mathbb{C}$ by

$$f_n(z) = z^n + c,$$

- where $n \geq 2$ is an integer, and
- $c \in \mathbb{C}$ is a complex parameter.



$f_{2,-0.12+0.75i}$



$f_{2,-0.15+i}$

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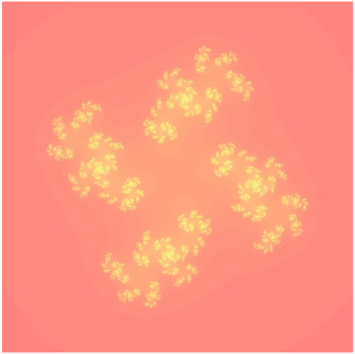

$f_{4,-0.12+0.75i}$

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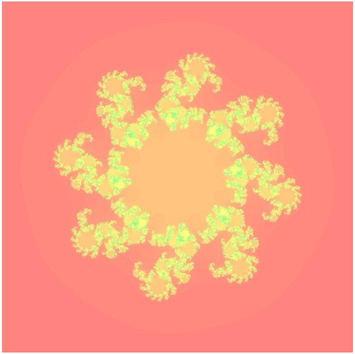
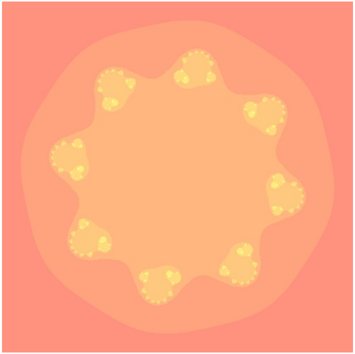


$f_{8,-0.12+0.75i}$



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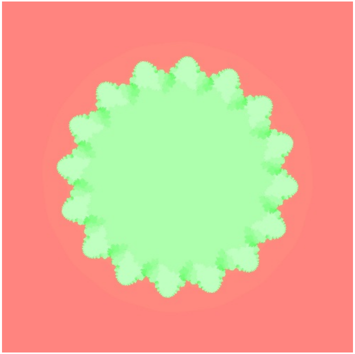
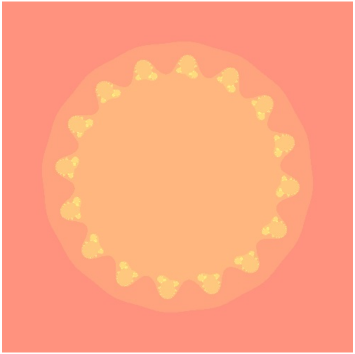


$f_{16,-0.12+0.75i}$



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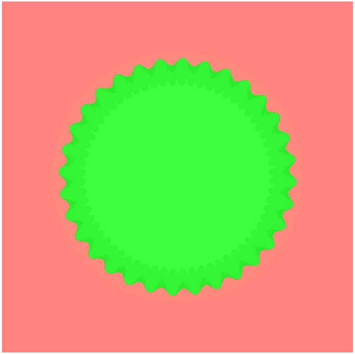
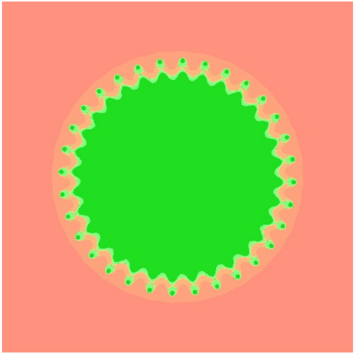


$f_{32,-0.12+0.75i}$



$f_{32,-0.15+i}$

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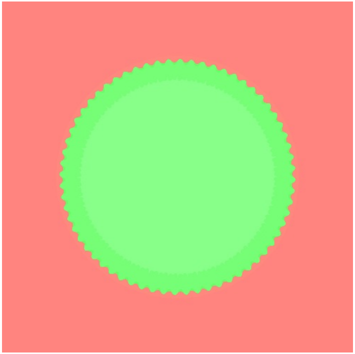
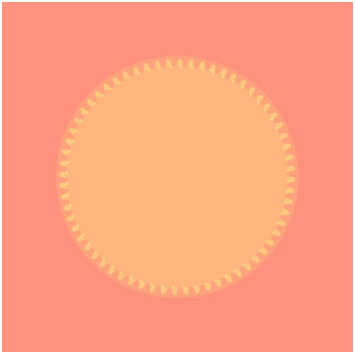


$f_{64,-0.12+0.75i}$



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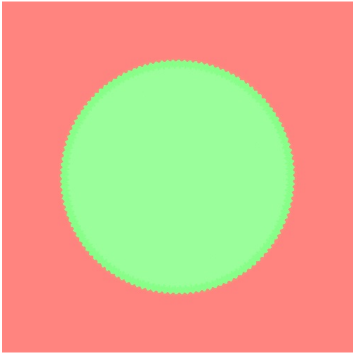
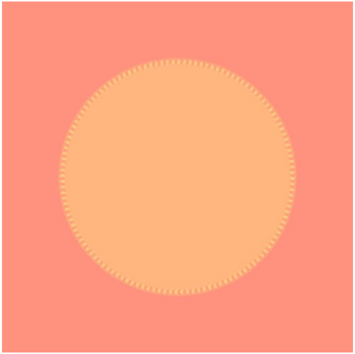


$f_{128,-0.12+0.75i}$



$f_{128,-0.15+i}$

Boyd-Schulz

$$f_{n,c}(z) = z^n + c$$

Theorem (Boyd-Schulz, 2012 [3])

Let $c \in \mathbb{C}$. Using the Hausdorff metric,

(1) *If $c \in \mathbb{C} \backslash \overline{\mathbb{D}}$, then $\lim_{n\to\infty} K(f_{n,c}) = S_0 = \{|z| = 1\}$.*

(2) *If $c \in \mathbb{D}$, then $\lim_{n\to\infty} K(f_{n,c}) = \overline{\mathbb{D}} = \{|z| \leq 1\}$.*

(3) *If $c \in S^1$, then if $\lim_{n\to\infty} K(f_{n,c})$ exists, it is contained in $\overline{\mathbb{D}}$.*

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(3) was further improved in [5] (2015).

A brief thread through history

2012 — [3] Boyd & Schulz: $f_n(z) = z^n + c$.

2015 — [5] Kaschner, Romero, & Simmons: $f_n(z) = z^2 + c$.

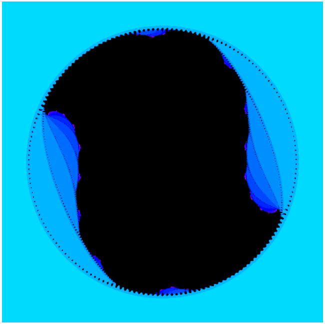
2020 — [4] Brame & Kaschner: $f_n(z) = z^n + q(z)$.

More geometric limits of Julia sets

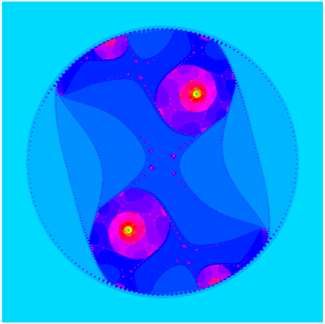
Let $f_n \colon \mathbb{C} \to \mathbb{C}$ by

$$f_n(z) = z^n + q(z),$$

- where $n \geq 2$ is an integer, and
- q is a fixed degree d polynomial.



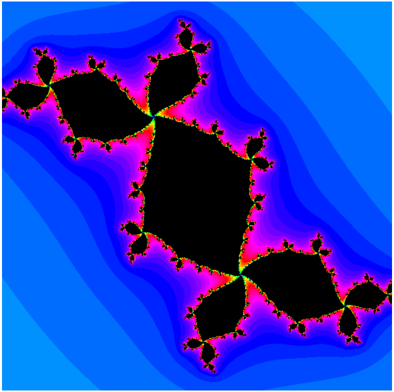
$f_{200, z^2+0.25+0.25i}$



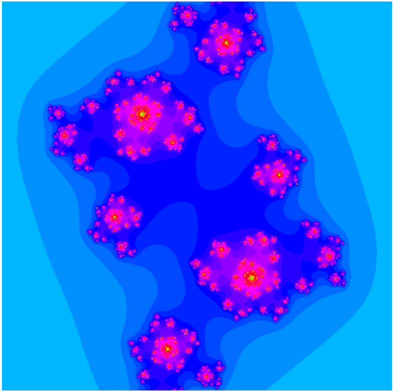
$f_{200, z^2+0.45+0.25i}$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 4$$



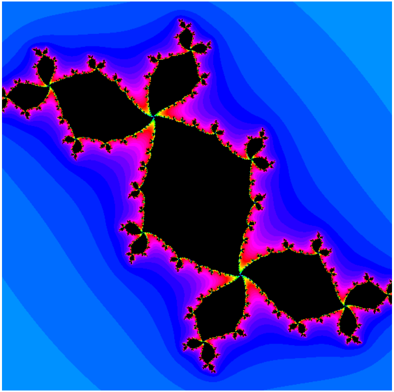
$K(q)$



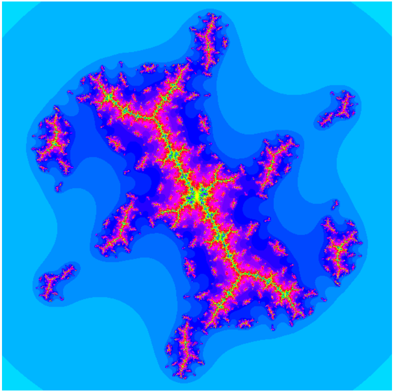
$K(f_{4,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 8$$



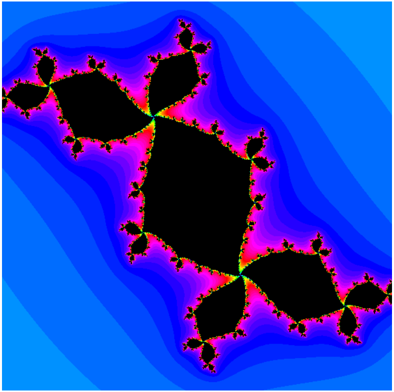
$K(q)$



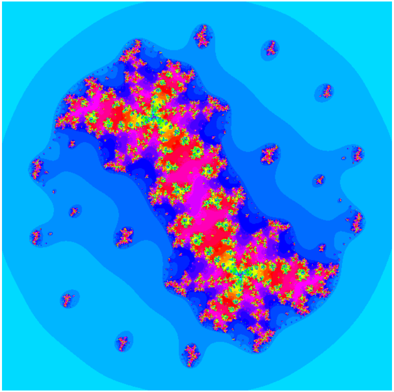
$K(f_{8,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 16$$



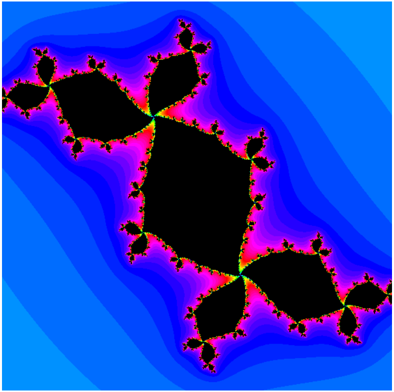
$K(q)$



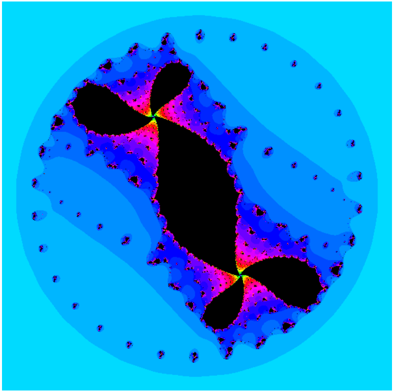
$K(f_{16,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 32$$



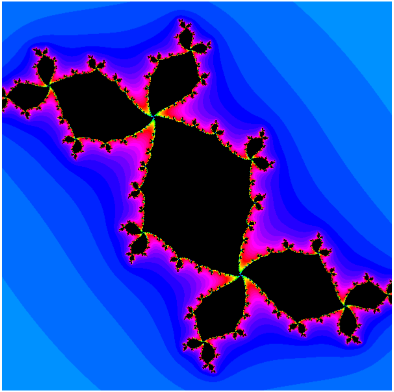
$K(q)$



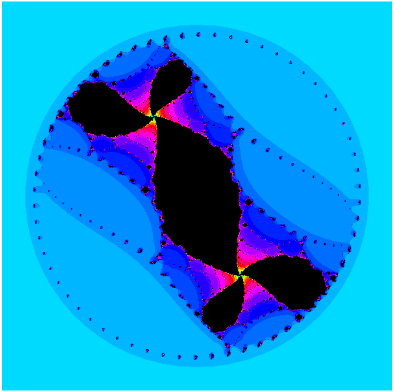
$K(f_{32,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 64$$



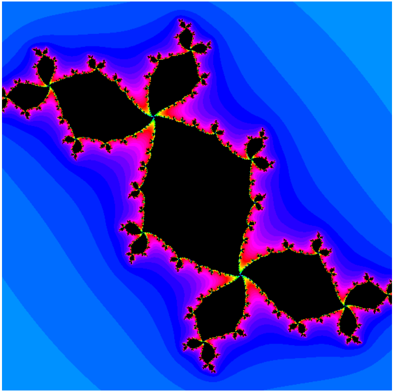
$K(q)$



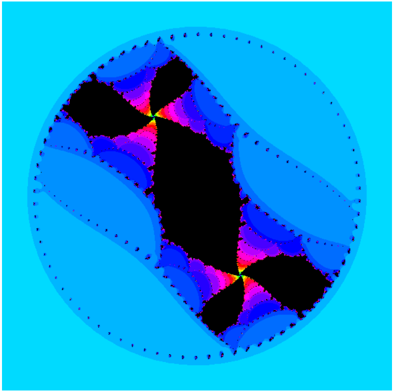
$K(f_{64,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 80$$



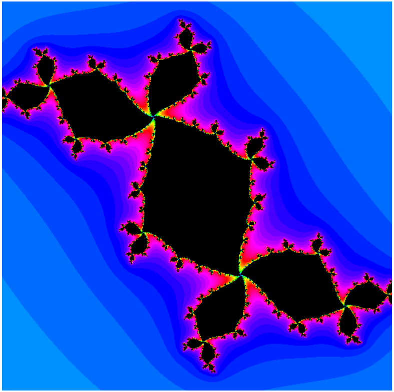
$K(q)$



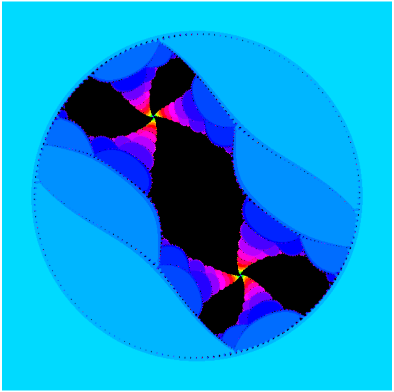
$K(f_{80,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 180$$



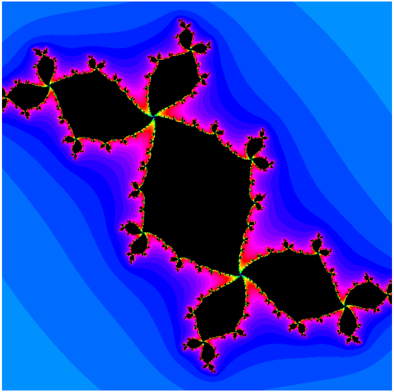
$K(q)$



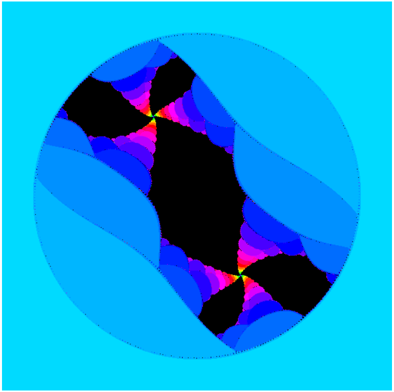
$K(f_{180,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 360$$



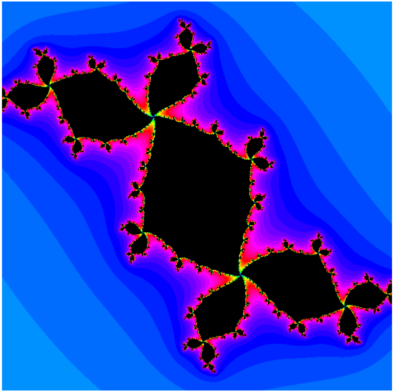
$K(q)$



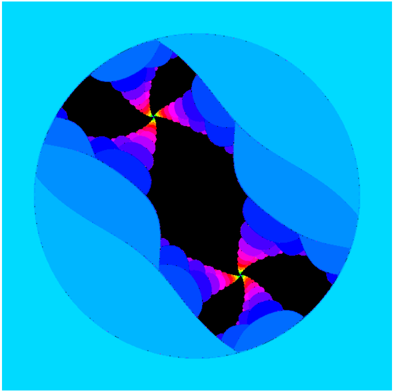
$K(f_{360,q})$

Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 720$$



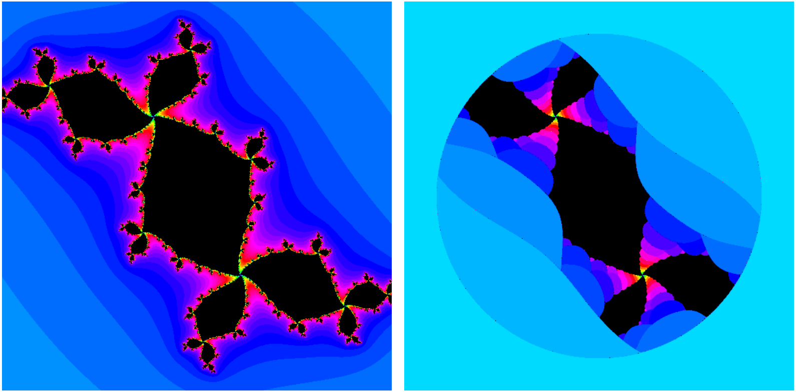
$K(q)$



$K(f_{720,q})$

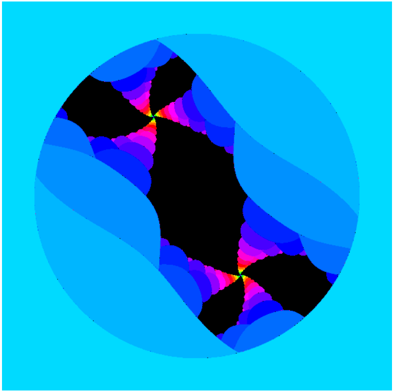
Rabbit in a cage

$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 1800$$



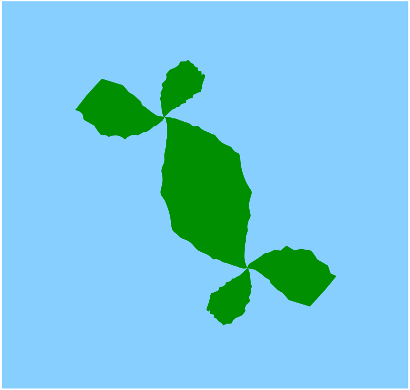
$K(q)$ $\qquad$ $K(f_{1800,q})$

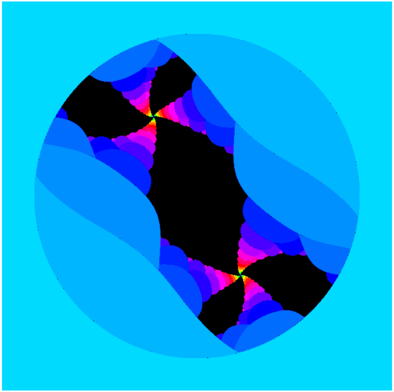
The limit set

The limit set

$$K_q = \bigcap_{i=0}^{\infty} q^{-i}(\bar{\mathbb{D}}) = \{z\colon q^i(z) \in \bar{\mathbb{D}}\ \forall i \geq 0\}$$





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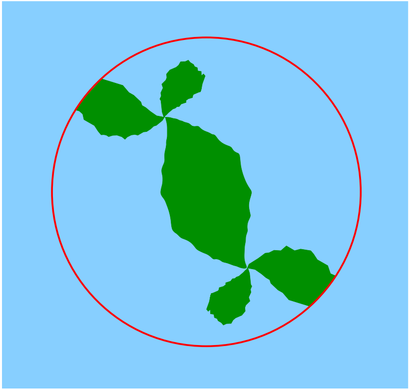
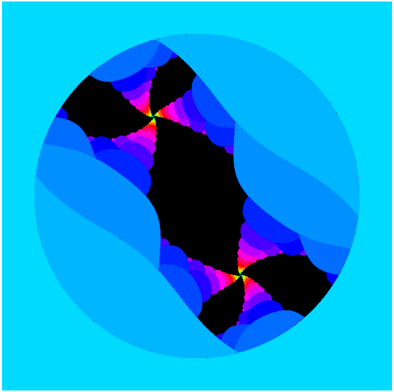
$$S_0 = \{z \colon |z| = 1\}$$





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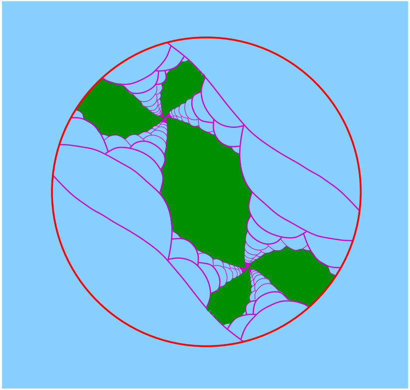
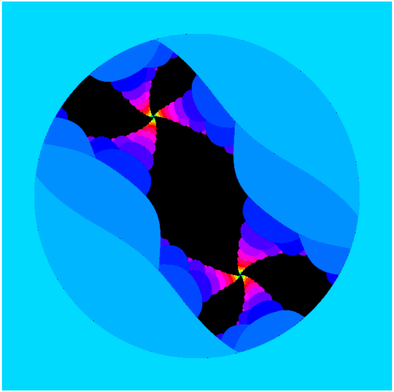
$$S_0 = \{z \colon |z| = 1\}$$

$$S_j = \{q^j(z) \in \partial\mathbb{D} \text{ and } q^i(z) \in \mathbb{D} \text{ for } i = 1, \ldots, j-1\}$$





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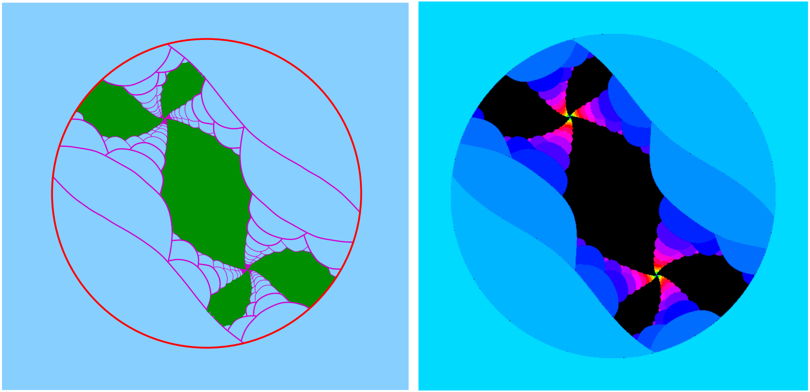

$$\lim_{n\to\infty} K(f_{n,q}) = K_q \cup \bigcup_{j\geq 0} S_j$$

Brame & Kaschner

$$f_n(z) = z^n + q(z)$$

Theorem (Brame-Kaschner, 2020 [4])

If $\deg q \geq 2$, *q is hyperbolic, and q has no attracting fixed points in* S_0, *then*

$$\lim_{n\to\infty} K(f_{n,q}) = K_q \cup \bigcup_{j\geq 0} S_j.$$

A brief thread through history

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2015 — [5] Kaschner, Romero, & Simmons: $f_n(z) = z^2 + c$.

2020 — [4] Brame & Kaschner: $f_n(z) = z^n + q(z)$.

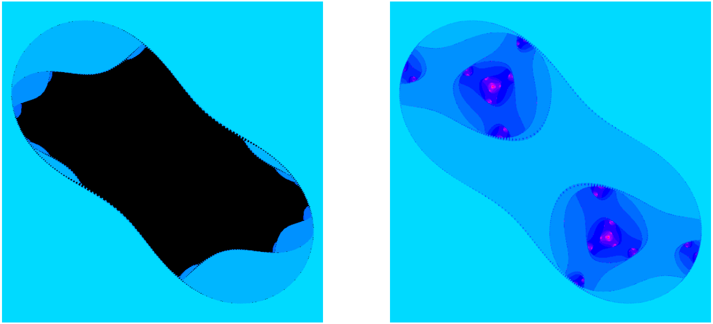
2023 — Kaschner, Kapiamba, & W.: $f_n(z) = (p(z))^n + q(z)$.

Even more geometric limits of Julia sets

Let $f_n\colon \mathbb{C} \to \mathbb{C}$ by

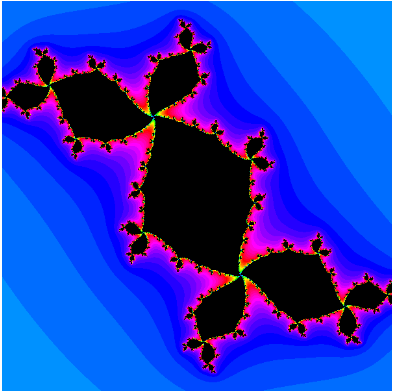
$$f_n(z) = (p(z))^n + q(z),$$

- where $n \geq 2$ is an integer, and
- p, q are fixed polynomials.

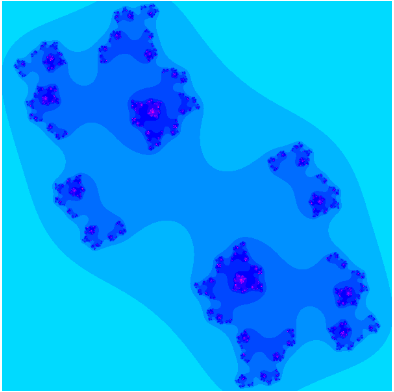


Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 4$$



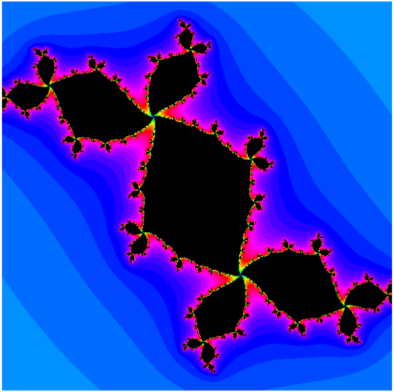
$K(q)$



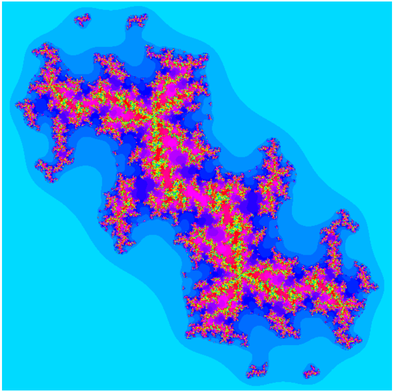
$K(f_4)$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 8$$



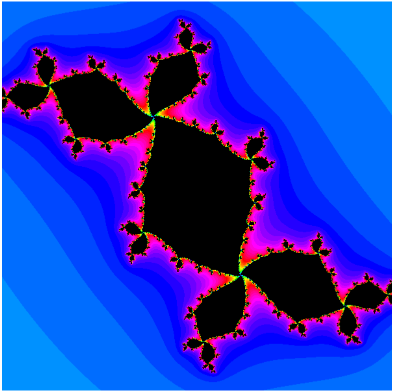
$K(q)$



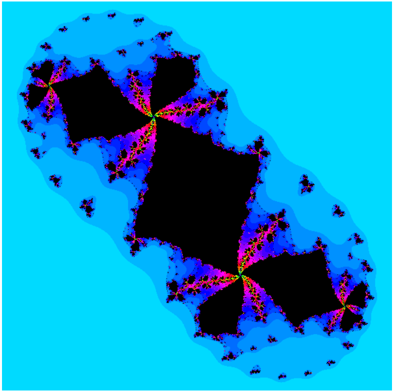
$K(f_8)$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 16$$



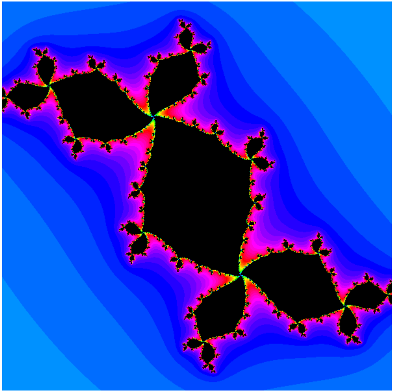
$K(q)$



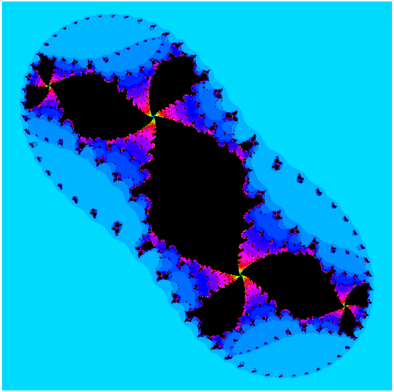
$K(f_{16})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 32$$



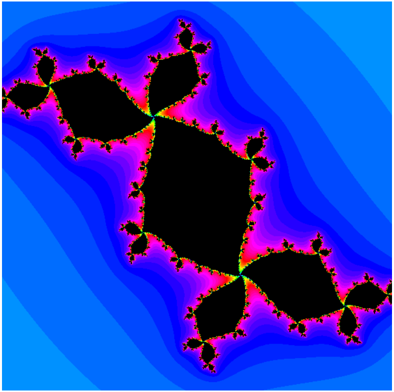
$K(q)$



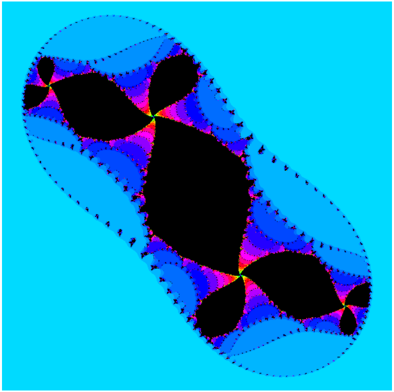
$K(f_{32})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 64$$



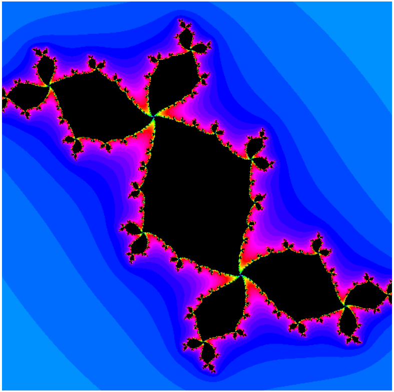
$K(q)$



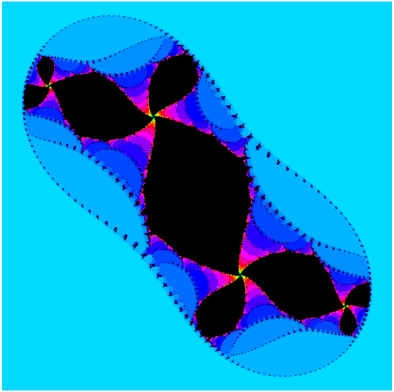
$K(f_{64})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 80$$



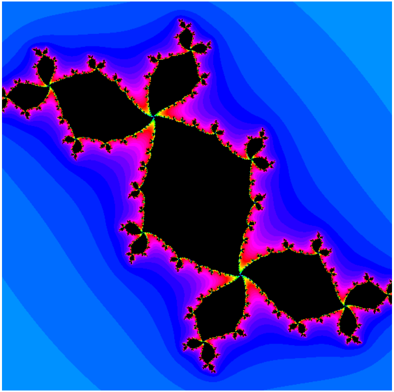
$K(q)$



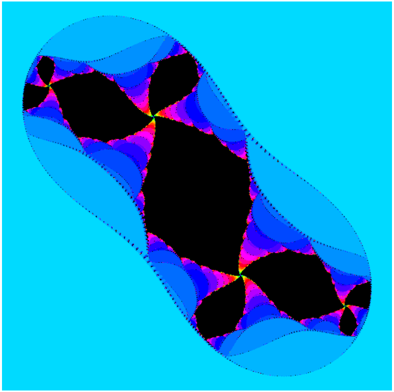
$K(f_{80})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 180$$



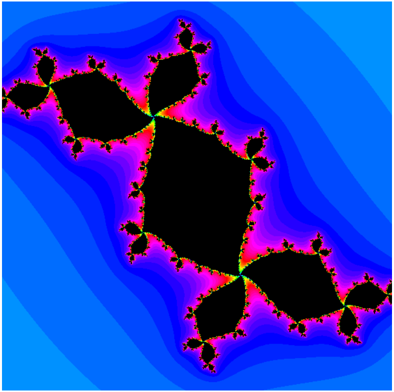
$K(q)$



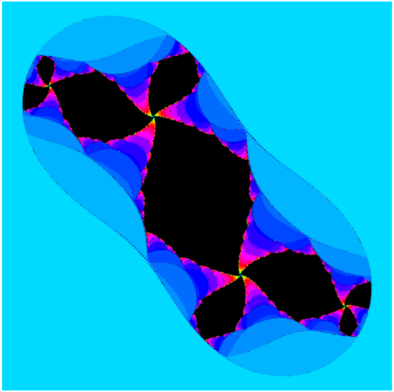
$K(f_{180})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 360$$



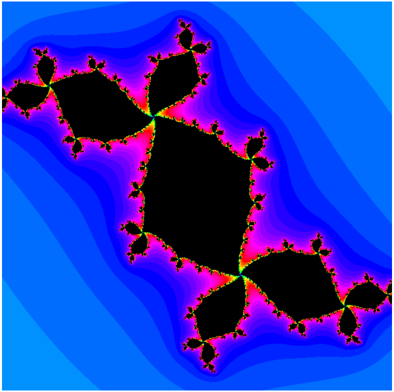
$K(q)$



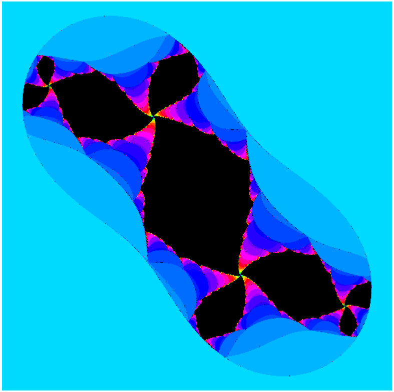
$K(f_{360})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 720$$



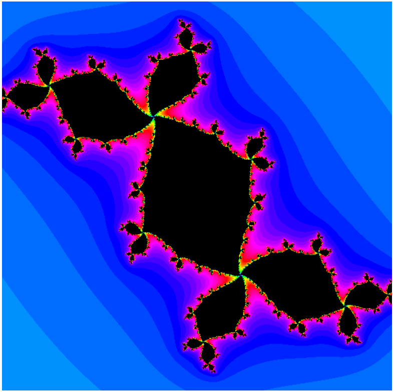
$K(q)$



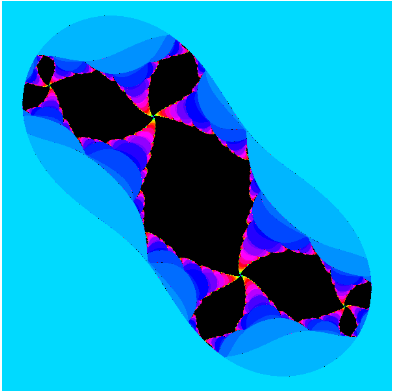
$K(f_{720})$

Into the Rabbitverse

$$p(z) = z^2 + 0.05 + 0.75i,$$
$$q(z) = z^2 - 0.1 + 0.75i,$$
$$n = 1800$$

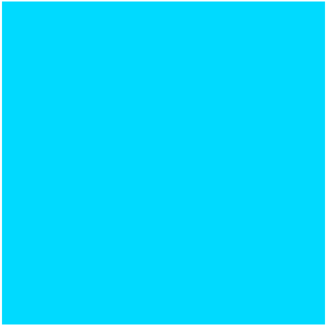


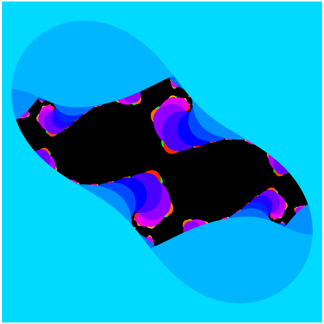
$K(q)$



$K(f_{1800})$

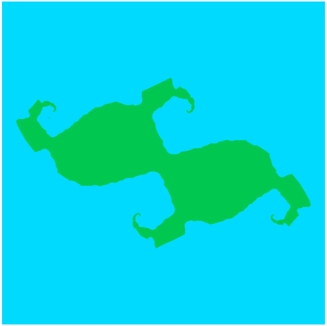
The trouble with Quibbles

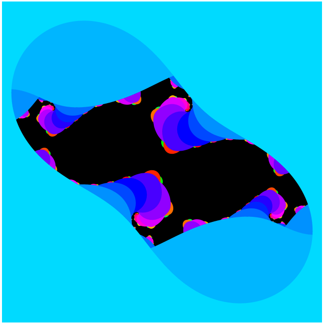




The trouble with Quibbles

$$K_q = \bigcap_{j=0}^{\infty} q^{-j}\left(p^{-1}(\bar{\mathbb{D}})\right)$$





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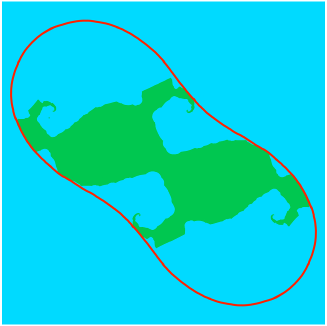
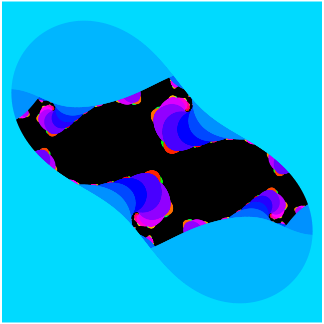
$$K_q = \bigcap_{j=0}^{\infty} q^{-j}\left(p^{-1}(\bar{\mathbb{D}})\right)$$

$$\mathcal{Q}_0 = \left\{p^{-1}(z) \colon |z| = 1\right\}$$





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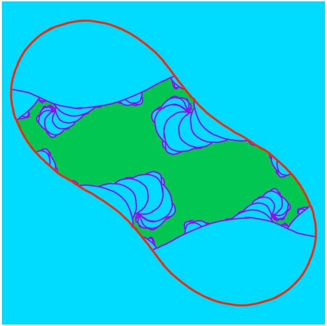

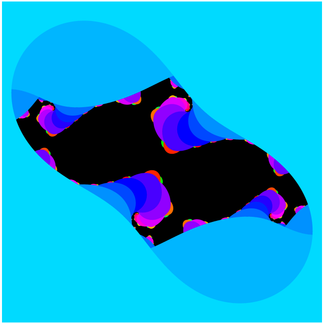

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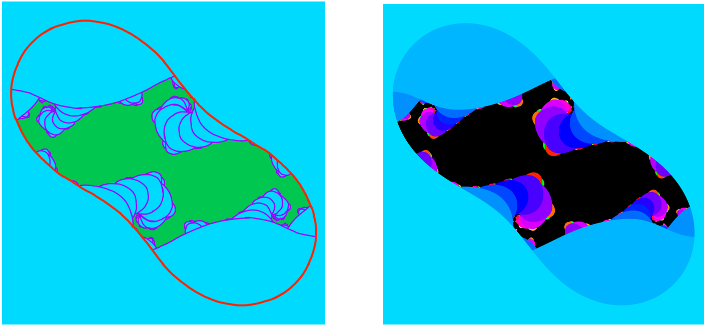


$$\lim_{n\to\infty} K(f_n) = K_q \cup \bigcup_{j\geq 0} \mathcal{Q}_j$$

Generalization

$$f_n(z) = (p(z))^n + q(z)$$

Theorem 1 (Kaschner, Kapiamba, & W.; 2023)

If p, q are polynomials with $\deg p, q \geq 2$, and q is hyperbolic with no attracting periodic points on $\partial p^{-1}(\overline{\mathbb{D}})$, then

$$\lim_{n\to\infty} K(f_{n,p,q}) = K_q \cup \bigcup_{j\geq 0} \mathcal{Q}_j$$

A brief thread through history... and the future

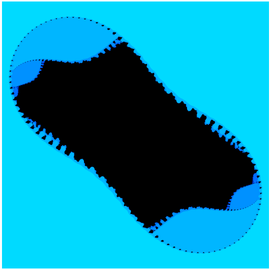
- **2012** — [3] Boyd & Schulz: $f_n(z) = z^n + c$.
- **2015** — [5] Kaschner & Romero & Simmons: $f_n(z) = z^2 + c$.
- **2020** — [4] Brame & Kaschner: $f_n(z) = z^n + q(z)$.
- **2023** — Kaschner, Kapiamba, & W.: $f_n(z) = (p(z))^n + q(z)$.
- **2024** — Kaschner, Kapiamba, & W.: $g_n(z) = p^n(z) + q(z)$.

Current work

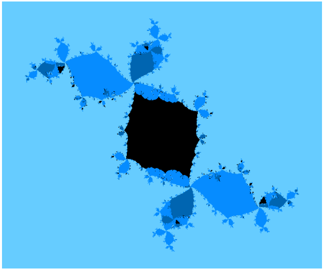
$$\underset{\text{powers}}{(p(z))^n} \neq \underset{\text{iterates}}{p^n(z)}$$

Behold, for

- $p(z) = z^2 - 0.1 + 0.75i$,
- $q(z) = z^2 - 0.1 + 0.2i$;
- $n = 51$;



$f_n = (p(z))^n + q(z)$



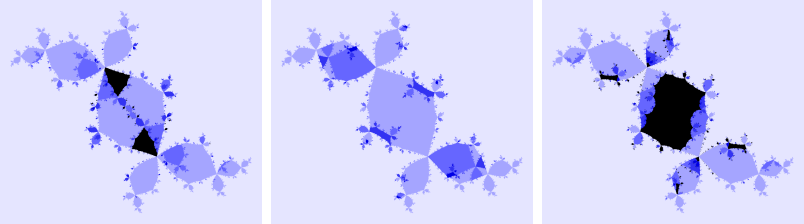
$g_n = p^n(z) + q(z)$

Immediate issues with subsequential limits

$$g_n(z) = p^n(z) + q(z)$$

$$p(z) = z^2 - 0.123 + 0.745i \quad q(z) = z^2 - 0.2 - 0.3i$$

$K(g_n)$ for $n = 49, 50, 51.$



Immediate issues with subsequential limits

$$g_n(z) = p^n(z) + q(z)$$

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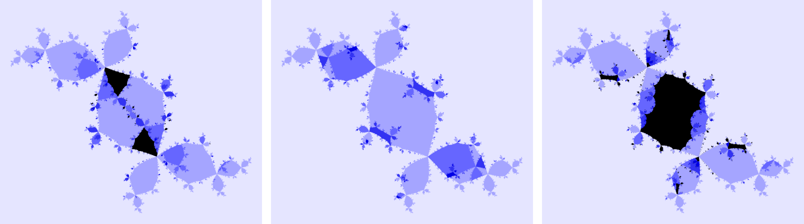
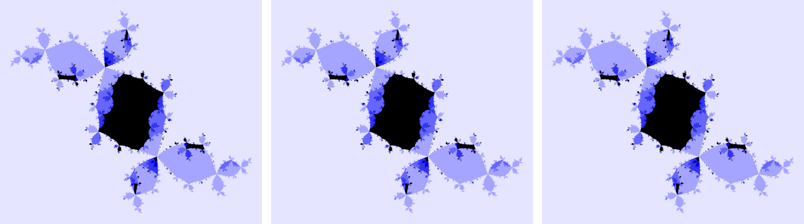
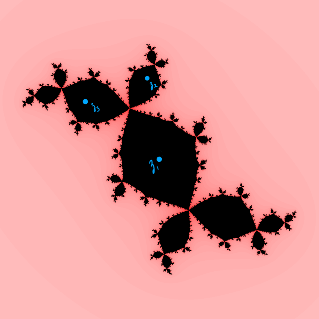




$K(g_n)$ for $n = 54, 57, 60.$

Escaping the Rabbitverse

- Suppose p is hyperbolic with periodic attracting cycle $z_1, z_2, \cdots, z_k$

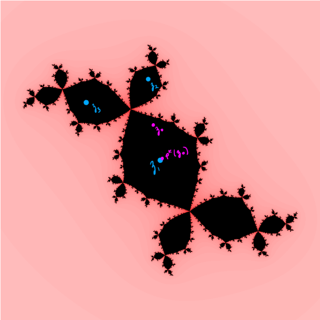


$K(p)$ for
$p(z) = z^2 - 0.123 + 0.745i$.

Escaping the Rabbitverse

- Suppose p is hyperbolic with periodic attracting cycle $z_1, z_2, \cdots, z_k$
- For each n, there exists some $\ell \in \{1, 2, \cdots, k\}$ such that

$$g_n(z) = p^{km+\ell}(z) + q(z)$$
$$\approx z_\ell + q(z)$$



$K(p)$ for
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Escaping the Rabbitverse

- Suppose p is hyperbolic with periodic attracting cycle $z_1, z_2, \cdots, z_k$
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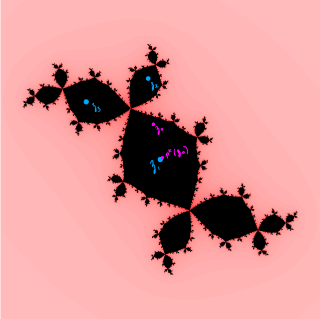
$$\begin{aligned} g_n(z) &= p^{km+\ell}(z) + q(z) \\ &\approx z_\ell + q(z) \end{aligned}$$

- Let $\mathcal{A}_\ell$ be the basin of attraction for z_ℓ for p^k, and $\mathcal{A} = \bigcup_{\ell=1}^{k} \mathcal{A}_\ell$



$K(p)$ for
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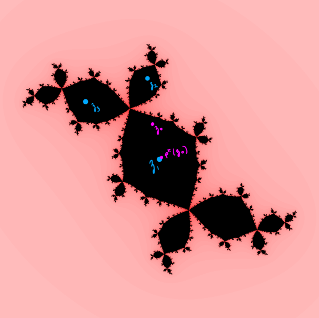
Escaping the Rabbitverse

- Suppose p is hyperbolic with periodic attracting cycle $z_1, z_2, \cdots, z_k$
- For each n, there exists some $\ell \in \{1, 2, \cdots, k\}$ such that

$$\begin{aligned} g_n(z) &= p^{km+\ell}(z) + q(z) \\ &\approx z_\ell + q(z) \end{aligned}$$

- Let $\mathcal{A}_\ell$ be the basin of attraction for z_ℓ for p^k, and $\mathcal{A} = \bigcup_{\ell=1}^{k} \mathcal{A}_\ell$
- Define $\hat{g}(z) : \mathcal{A} \to \mathbb{C}$ via

$$\hat{g}(z) = \begin{cases} q(z) + z_1, & \text{for } z \in \mathcal{A}_1 \\ \vdots & \\ q(z) + z_k, & \text{for } z \in \mathcal{A}_k. \end{cases}$$



$K(p)$ for
$p(z) = z^2 - 0.123 + 0.745i$.

Current conjecture

Suppose p is hyperbolic and $q(z) + z_\ell$ is hyperbolic for each $\ell \in \{1, 2, \ldots, k\}$. For some fixed ℓ, define the subsequence $n_m = \ell + mk$. Then

$$\lim_{m\to\infty} K(g_{n_m}) = \bigcap_{j=0}^{\infty} \hat{g}^{-j}(p^\ell(\mathcal{A})) \cup \bigcup_{j=0}^{\infty} \mathcal{J}_j$$

where

$$\mathcal{J}_j = \{z : \hat{g}^j(z) \in J(p) \text{ and } \hat{g}^\kappa(z) \in \mathcal{A} \text{ for } \kappa = 1, 2, \ldots, j-1\}$$

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Scott Kaschner
(Butler)

Alex Kapiamba
(Brown)

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THANK YOU!

Eleanor Waiss

kwaiss@butler.edu

Geometric Limits of Julia Sets

Butler University