MA334 PRE-COURSE REVIEW "EXAM"

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Note: I do not guarantee the accuracy of the work herein. I am but a student seeing the same material you are for the first time. I try to fully motivate and answer everything, but some pieces may be omitted. If you have questions, please refer to the first sentence of this paragraph.

(1) Find the derivative with respect to t of $x(t) = (\tan(\alpha t^2 + 5)) \ln(\sqrt{t})$.

Note: $x(t) = a(b(t)) \cdot c(d(t))$ for

$$a(t) = \tan t \qquad b(t) = \alpha t^2 + 5 \qquad c(t) = \ln t \qquad d(t) = \sqrt{t}$$
$$\frac{\mathrm{d}a}{\mathrm{d}t} = \sec^2 t \qquad \frac{\mathrm{d}b}{\mathrm{d}t} = 2\alpha t \qquad \frac{\mathrm{d}c}{\mathrm{d}t} = \frac{1}{t} \qquad \frac{\mathrm{d}d}{\mathrm{d}t} = \frac{1}{2\sqrt{t}}$$

thus, after applying the chain and product rules, we have that

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \sec^2(\alpha t^2 + 5)(2\alpha t)(\frac{1}{2}\ln t) + \tan(\alpha t^2 + 5)\frac{1}{t} \cdot \frac{1}{2\sqrt{t}} \\ &= \alpha t \sec^2(\alpha t^2 + 5) \ln t + \frac{1}{2t^{3/2}} \tan(\alpha t^2 + 5).\end{aligned}$$

(2) Evaluate $f'(0)$ if $f(x) = \frac{321 \ln(x+1)}{x \sin x + 1}$.

Note that we can rewrite f as $321 \ln(x+1)\,(x \sin x + 1)^{-1}$, thus after repeated application of the chain and product rules, we have that

$$\begin{aligned}f'(x) &= \frac{321}{x+1}\,(x \sin x + 1)^{-1} + 321 \ln(x+1)(-1)(x \sin x + 1)^{-2}(\sin x + x \cos x) \\ &= \frac{321}{(x+1)(x \sin x + 1)} - \frac{321 \ln(x+1)(\sin x + x \cos x)}{(x \sin x + 1)^2}\end{aligned}$$

after which evaluating f' at $x = 0$ returns 321.

(3) Find a function f and a number a such that $6 + \int_a^x \frac{f(t)}{t^2}\mathrm{d}t = 2\sqrt{x}$ for all $x > 0$.

In young Anakin Skywalker's voice: "I'll try differentiating, that's a good trick!".

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$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[6+\int_a^x \frac{f(t)}{t^2}\mathrm{d}t\right] &= \left[2\sqrt{t}\right] \\
\frac{f(x)}{x^2} &= \frac{1}{\sqrt{x}} \\
f(x) &= x^{3/2}
\end{aligned}$$

and substituting this value back in...

$$\begin{aligned}
6+\int_a^x \frac{t^{3/2}}{t^2}\mathrm{d}t &= 2\sqrt{x} \\
6+2t^{1/2}\Big|_a^x &= 2\sqrt{x} \\
6+2\sqrt{x}-2\sqrt{a} &= 2\sqrt{x} \\
a &= 9.
\end{aligned}$$

(4) Calculate $\int \frac{x+2}{(x-1)(x+3)}\mathrm{d}x$.

Using partial fraction decomposition, we know that $x+2 = A(x-3)+B(x-1)$ for the correct choice of A and B. Using the specific values of $x = -1, 3$, we find that $A = \frac{-3}{2}$ and that $B = \frac{5}{2}$. Thus, from the linearity of integrals, we have that

$$\int f\mathrm{d}x = \frac{5}{2}\ln(x+3) - \frac{3}{2}\ln(x-1) + c$$

(5) Calculate $\int t\sin(2t)\mathrm{d}t$.

From integration by parts or via tabular integration, we have that

$$\int f\mathrm{d}t = \frac{-t}{2}\cos 2t + \frac{1}{4}\sin 2t + c$$

(6) Find the area in the first quadrant enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

Rearranging, we have that...

$$\begin{aligned}
\left(\frac{y}{b}\right)^2 &= 1-\left(\frac{x}{a}\right)^2 \\
y &= b\sqrt{1-\left(\frac{x}{a}\right)^2} \\
&= \frac{b}{a}\sqrt{a^2-x^2}.
\end{aligned}$$

Integrating to compute area...

$$\begin{aligned}
A &= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\mathrm{d}x \\
&= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\mathrm{d}x
\end{aligned}$$

...and using the substitutions $x = a\sin t$, $\mathrm{d}x = a\cos t\mathrm{d}t$, and correcting the bounds of integration...

$$
\begin{aligned}
A &= \frac{b}{a}\int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2 t}a\cos t\mathrm{d}t \\
&= ab\int_0^{\pi/2} \sqrt{1-\sin^2 t}\cos t\mathrm{d}t \\
&= ab\int_0^{\pi/2} \cos^2 t\mathrm{d}t \\
&= ab\int_0^{\pi/2} (\frac{1}{2}\cos 2t + \frac{1}{2})\mathrm{d}t \\
&= ab\left(\frac{1}{4}\sin 2t + \frac{1}{2}t\Big|_0^{\pi/2}\right) \\
&= \frac{1}{4}\pi ab
\end{aligned}
$$

(7) Write the product of complex numbers $(1+i)(\sqrt{3}-i)$ in polar form.

Note, that for $z \in \mathbb{C}$, we have that $z = x + iy = \sqrt{x^2+y^2}e^{i\arctan(\frac{y}{x})}$. Thus,

$$(1+i)(\sqrt{3}-i) = (\sqrt{2}e^{i\pi/4})(2e^{-i\pi/6}) = 2\sqrt{2}e^{i\pi/12}.$$