

subject

date
3 Dec 2024

keywords

topic

Test 2 Debrief
Mean ~66

Ch 7 Series Solns
Ch 8 Nonlinear Nightmare

Recall: Power Series $\sum_{n=0}^{\infty} c_n (x-a)^n$

infinite degree polynomial
Centered at a
coefficients in $\mathbb{R}$
Converges on some "interval of convergence"

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad I = (-1,1)$$

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad I = \mathbb{R}$$

Give polynomial approximation to any piecewise continuous (Stone-Weierstrass)

A fxn is analytic iff $\exists$ open $I \ni x_0$ on which $f\big|_I = \sum_{k=0}^{\infty} c_k (x-x_0)^k\Big|_I$

Taylor's theorem (building a power series using ∞ derivatives) is actually a weaker condition than analyticity

Recall $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \rightsquigarrow y'' + p(x)y' + q(x)y = 0$

2nd order, linear, homog
Unless p,q constants, non-autonomous

x is ordinary if p,q are analytic in x
x is singular if not

Theorem

If x_0 is ordinary, then $\exists!$ 2 linearly independent solutions $y = \sum_{n=0}^{\infty} c_n (x-x_0)^n$ converging on $|x-x_0| < R$
where R = distance from x_0 to nearest singular point

Note, x_0 needs to be such that $R \neq 0$, i.e. x_0 is not singular

Ex

$4y'' + y = 0$ Could $y = e^{rt}$ this, but we won't!

$\rightsquigarrow y'' + \frac{1}{4}y = 0$

$p(x) = 0$, $q(x) = \frac{1}{4}$, both analytic on $\mathbb{R}$ (or $\mathbb{C}$)

Let $x_0 = 0$, guess $y = \sum_{n=0}^{\infty} c_n x^n$. Plug in guess and solve!

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\rightsquigarrow 4\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n$$

Reindex!

$$= \sum_{k=0}^{\infty} \left(4(k+2)(k+1)c_{k+2} + c_k\right) x^k \stackrel{\cdot}{=} 0$$

We can only do this because p,q analytic!

$$\implies 4(k+2)(k+1)c_{k+2} + c_k = 0 \quad \forall k \in \mathbb{Z}_{\geq 0}$$

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$\rightsquigarrow\ c_{k+2} = \dfrac{-c_k}{4(k+2)(k+1)}$ Recursive relation/ difference equation

$k=0 \rightsquigarrow c_2 = \dfrac{-c_0}{4\cdot 2} = \dfrac{-c_0}{4^1}\cdot\dfrac{1}{2!}$

$k=1 \rightsquigarrow c_3 = \dfrac{-c_1}{4\cdot 3\cdot 2} = \dfrac{-c_1}{4^1}\cdot\dfrac{1}{3!}$

$k=2 \rightsquigarrow c_4 = \dfrac{-c_2}{4\cdot 4\cdot 3} = \dfrac{c_0}{4^3\cdot 3\cdot 2} = \dfrac{c_0}{4^2}\cdot\dfrac{1}{4!}$

$k=3 \rightsquigarrow c_5 = \dfrac{-c_3}{4\cdot 5\cdot 4} = \dfrac{c_1}{4^3\cdot 5\cdot 3\cdot 2} = \dfrac{c_1}{4^2}\cdot\dfrac{1}{5!}$

$$y = \sum_{k=0}^{\infty} c_k x^k = c_0\left(1 - \frac{1}{4\cdot 2!}x^2 + \frac{1}{4^2\cdot 4!} - \frac{1}{4^3\cdot 6!}x^6 + \cdots\right)$$
$$+ c_1\left(x - \frac{1}{4\cdot 3!}x^3 + \frac{1}{4^2\cdot 5!}x^5 - \frac{1}{4^3\cdot 7!}x^7 + \cdots\right)$$
$$= c_0\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k(2k)!}x^{2k}\right] + c_1\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(2k+1)!}x^{2k+1}\right]$$