Math 334 – Differential Equations

Notes on *Notes on Diffy Qs, Differential Equations for Engineers*, Jiří Lebl

Sections 0.2–3

0 Introduction

0.2 Introduction to differential equations

> **Differential Equation.** A *differential equation* is an equation with a derivative in it.

Example 1.

$$\frac{d^2x}{dt^2} + x\frac{dx}{dt} = 6t$$

- What is x? dependent variable
- What is t? independent variable

$\frac{dy}{dx}$ vs y' vs $\dot{y}$

$\frac{d^2y}{dx^2}$ vs y'' vs $\ddot{y}$

$$y'' + xy' = 6x$$

- What's the difference between this differential equation and the one before it?

Higher order! Also, old (calc 1)

> **Solution.** A *solution* for a differential equation is a function that satisfies the equation (makes the equation true). Any single solution is called a *particular solution.* The set of all solutions is called the *general solution.*

Example 2. The differential equation

$$y' = 3x^2$$

is very boring. Why?

- A particular solution is specified constants
- The general solution is unknown constants (of integration) i.e. all solutions

Why is the equation in Example 1 *much* harder to solve?

ODE: single independent variable

PDE: multiple independent variables

We will learn when and how differential equations can be solved analytically (almost never).

Barring that, we will learn how to approximate and use solutions.

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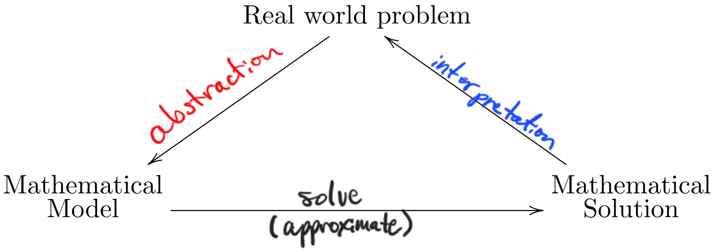
[1] We should probably come up with some more specific terminology.

Who cares about these things? Right.



Example 3. $P(t) = Ce^{kt}$ is the general solution for $\dfrac{dP}{dt} = kP$. Check this.

$$\frac{dP}{dt} = Cke^{kt} = k\left(Ce^{kt}\right) = kP \checkmark$$

- What does this have to do with the flow chart above?

Example 4. Show $y = \cosh t = \frac{1}{2}(e^t + e^{-t})$ is a particular solution for $\dfrac{d^2y}{dt^2} - y = 0$ on the interval $(-\infty, \infty)$.

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(e^t + (-(-e^{-t}))\right) = \frac{1}{2}\left(e^t + e^{-t}\right) = y \Rightarrow \frac{d^2y}{dt^2} = y \checkmark$$

Example 5. For what values of r is $y = e^{rt}$ a solution for $y'' + y' - 6y = 0$?

$$r^2e^{rt} + re^{rt} - 6e^{rt} = 0$$

$$(r-2)(r+3) = 0$$

$$\boxed{r = 2, -3}$$

0.3 Classification of differential equations

Here is a terrible wall of definitions. Enjoy!

Order. The *order* of a differential equation is the order of the highest derivative that appears in the equation. More specifically,[a] a differential equation of order n is of the form

$$F\left(t, x(t), \frac{dx}{dt}, \dots, \frac{d^n x}{dt^n}\right) = 0,$$

where F is a function.

---

[a] or is this more generally?

Autonomous. If F (as above) is independent of t, the differential equation is called *autonomous*. Otherwise, it is called *nonautonomous*.

the independent variable

Linear and homogeneous. A differential equation of order n is called *linear* if it is of the form

$$F\left(t, x(t), \frac{dx}{dt}, \dots, \frac{d^n x}{dt^n}\right) = a_n(t)\frac{d^n x}{dt^n} + \cdots + a_1\frac{dx}{dt} + a_0 x + b(t),$$

where the a_i's and b are all functions of t. If $b(t) = 0$, then the differential equation is called *homogeneous*; otherwise, it is called *nonhomogeneous*.

"What is all this madness?" you may ask. Well, different classifications of differential equations require different techniques and strategies.

Example 6. Classify the following differential equations:

- $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + 2y - \sin t = 0$

  Non-autonomous : not indep of t
  Linear : l.c. of $\frac{d^n x}{dt^n}$
  2^{nd} order
  non homogenous : $h(t) \neq 0$

- $y'' + y\underline{y'} = 0$

  2^{nd} order
  Autonomous
  non linear

Eg: $y' = y^x$ Problem w/ integrating directly

Eg $y' = xe^x$ or $\frac{dy}{dx} = xe^x$

$$Y = \int xe^x dx = \boxed{xe^x - e^x + c}$$

$$= e^x(x-1) + c$$

General Solution

+	x	e^x
−	1	e^x
+	0	e^x

↳ initial condition
$Y_0 = Y(X_0)$ to identify a particular solution

$Y(0) = 0$
$\rightarrow e^0(0-1) + c = 0 \rightarrow c = 1$

An ODE w/
initial conditions
is an Initial Value
Problem (IVP)

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Sections 1.1–2

1 First order ODEs

In case no one mentioned it, and *ODE* is an ordinary differential equation, which is just a differential equation with no partial derivatives (those are called PDEs). The word "ordinary" is just used to let you know that since there are no partial derivatives, you won't have to do anything too silly. While this course deals exclusively in ODEs, we maintain the right to do silly things.

1.1 Integrals as solutions

Which is easier to solve?

- $\dfrac{dy}{dx} = f(x, y)$
- $\dfrac{dy}{dx} = f(x)$

Why?

Example 1. Solve $y' = xe^x$. What do you need to identify a single particular solution?

Example 2. Solve $y' = xe^x$, $y(0) = 0$.

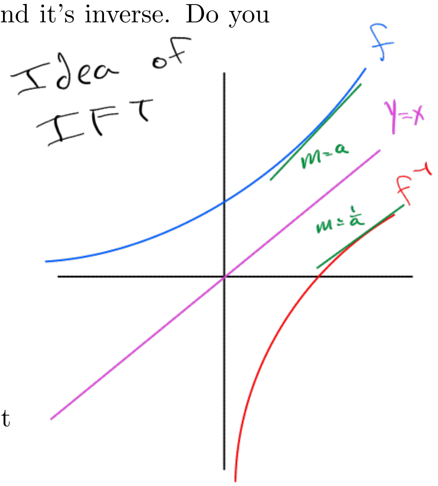
IVP. An *IVP*, or *initial value problem*, is an ODE with enough initial conditions to identify a single particular solution.

Can we solve $\frac{dy}{dx} = f(y)$? Why is this harder?

Here's a fun fact from Calculus 1 that will help:

> **Inverse Function Theorem.** If $y(x)$ is continuously differentiable and has a nonzero derivative at x_0, then
> $$\left(y^{-1}\right)'(y(x_0)) = \frac{1}{y'(x_0)}.$$
> That is, the derivative of the inverse at $y(x_0)$ is the reciprocal of the derivative at x_0.

This is a really neat theorem. Draw the graph of a nonlinear one-to-one function and it's inverse. Do you see why this theorem is true?



Don't forget that $\frac{dy}{dx} = f(y)$. When $x(y) = y^{-1}$ is differentiable, we have

Then from the Inverse Function Theorem, we know that

(when y is continuously differentiable and has a nonzero derivative). Now we can just

.

Example 3 (Exercise 1.1.6)**.** Solve $y' = (y-1)(y+1)$, $y(0) = 3$.

$\frac{dy}{dx}$

Not integrable

See next page for Sol'n

$$\frac{dx}{dy} = (y^{-1})' \quad \underline{\text{and}} \quad \frac{1}{y'} = \frac{1}{f(y)}$$

(not precise, but ignore it)

$$(y' = f(y))$$

IFT

$$\frac{dx}{dy} = \frac{1}{f(y)}$$

$$\frac{dy}{dx} = (y-1)(y+1)$$

$$\frac{dx}{dy} = \frac{1}{(y-1)(y+1)}$$

$$\int \frac{dx}{dy}\,dy = \int \frac{1}{(y-1)(y+1)}\,dy$$

$$\frac{1}{2}\int \frac{1}{y-1}\,\partial y - \frac{1}{2}\int \frac{1}{y+1}\,\partial y$$

$$= \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| + c$$

Solve for y

$$1 = A(y+1) + B(y-1)$$

$$B = \frac{-1}{2} \quad A = \frac{1}{2}$$

$$2x - 2c = \ln|y+1| - \ln|y-1|$$

$$2x - 2c = \ln\left|\frac{y+1}{y-1}\right|$$

$$e^{2x-2c} = \frac{y+1}{y-1}$$

$$ye^{2x-2c} - e^{2x-2c} = y+1$$

$$\boxed{y = \frac{1 + e^{2x}\cancel{e^{-2c}}\,C_0}{e^{2x}\underset{C_0}{\cancel{e^{-2c}}} - 1}}$$

General Solution

$$e^{-2c} = C_0$$

$$y = \frac{1 + 2e^{2x}}{2e^{2x} - 1}$$ particular solution

$$y(0) = 3 \rightarrow 3 = \frac{1 + C_0}{C_0 - 1} \rightarrow 3C_0 - 3 = 1 + C_0 \rightarrow C_0 = 2$$

$$y' = (y-1)(y+1)\ ;\ y(0) = 3$$

$$\|$$

$$\frac{dy}{dx} \xrightarrow[\text{Theorem}]{\text{Inverse Function}} \frac{dx}{dy} = \frac{1}{(y-1)(y+1)} \overset{\text{Partial Fractions}}{=} \frac{A}{y-1} + \frac{B}{y+1}$$

$$\Rightarrow\ 1 = A(y+1) + B(y-1)$$

$$y \doteq 1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$y \doteq -1 \Rightarrow 1 = -2B \Rightarrow B = \frac{-1}{2}$$

$$\Rightarrow\ x = \int \frac{dx}{dy}\,dy = \int \frac{1}{(y-1)(y+1)}\,dy = \frac{1}{2}\int \frac{1}{y-1}\,dy - \frac{1}{2}\int \frac{1}{y+1}\,dy$$

$$= \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| + c$$

$$\Rightarrow\ 2x - 2c = \ln\left|\frac{y-1}{y+1}\right|$$

$$\Rightarrow\ e^{2x}\underbrace{e^{-2c}}_{C_0} = \frac{y-1}{y+1} \Rightarrow C_0 e^{2x} y + C_0 e^{2x} = y - 1$$

$$\Rightarrow\ \left(C_0 e^{2x} - 1\right) y = -\left(1 + C_0 e^{2x}\right) \Rightarrow y = \frac{1 + C_0 e^{2x}}{1 - C_0 e^{2x}}$$

$$\underline{y(0) = 3} \Rightarrow 3 = \frac{1 + C_0 e^{2(0)}}{1 - C_0 e^{2(0)}}$$

$$\Rightarrow\ 3 - 3C_0 = 1 + C_0 \Rightarrow 2 = 4C_0 \Rightarrow C_0 = \frac{1}{2}$$

Check: $\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = {}^{3/2}\!/_{1/2} = 3$ ✓

$$\Rightarrow\ y = \frac{1 + \frac{1}{2}e^{2x}}{1 - \frac{1}{2}e^{2x}}\left(\frac{2}{2}\right) = \boxed{\frac{2 + e^{2x}}{2 - e^{2x}}}$$

1.2 Slope fields

Recall that, in general, first order equations are of the form

$$y' = f(x, y),$$

where f is any function you like, depending on *both* x and y. If f depends on just one of these variables, we saw in the last section that you can just integrate to solve.

What does the equation $y' = f(x, y)$ mean? It takes x and y values and assigns (by f) a value to y', often interpreted as *slope*. That is,

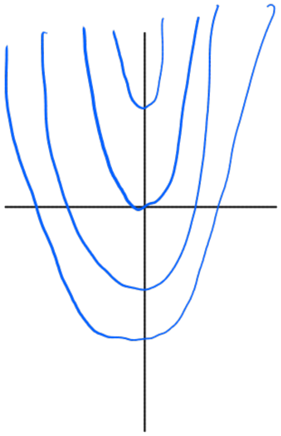
$\frac{dy}{dx} = f(x,y)$

We can graph this!

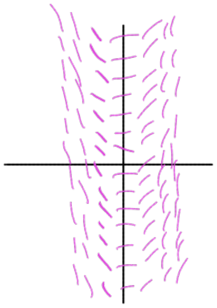
Example 4. Let $y' = 2x$. Plot the slope field by hand and find the general solution. Compare them.

$Y = x^2 + C$ as general solution

ODE relate slope to "indep. and dep fxn values"



etc



Vector Field

Google "bluffton slope field" and plot a slope field by way of internet.

Example 5. Plot a slope field (via computer) for $y' = x/y$. Beware computers.

What's wrong here?

Problems!
infinite slope?
Multiple solns?

Example 6. Plot a slope field (via computer) for $y' = 2\sqrt{|y|}$. Beware intuition.

What's wrong here?

Given a problem, there are two basic questions:

1.

2.

$\frac{dy}{dx} = f(x,y)$ is ODE

Picard's Theorem.[a] If $f(x,y)$ is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0), then a solution to the IVP

$$y' = f(x,y), \quad y(x_0) = x_0$$

exists near x_0 and is unique.

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[a] Also commonly referred to as the Fundamental Theorem of Existence and Uniqueness (FEU)

Example 7. $x' = x^{1/3}$, $x(0) = 0$ is a sufficiently simple-looking IVP, right? Show $x = 0$ is a solution, and for any nonnegative real α,

$$x(t) = \begin{cases} \left(\frac{2}{3}t\right)^{3/2}, & |t| < \alpha \\ 0, & t \leq -\alpha, \alpha \leq t \end{cases}$$

is also a solution. There are an uncountable number of solutions to this IVP.

What is happening here?

$f(t,x) = x^{1/3}$ $\qquad \frac{\partial f}{\partial x} = \frac{1}{3} x^{-2/3}$ ← not defined @ $x=0$, which is the initial condition

Example 8. Show $y' = 1 + y^2$, $y(0) = 0$ has a unique solution $y = \tan x$ on $(-\pi/2, \pi/2)$.

$\frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x$ ✓

$y(0) = \tan(0) = 0$ ✓

} soln to IVP

Picard's Theorem

$f(x,y) = 1 + y^2$

$\frac{\partial f}{\partial y} = 2y$ Continuous everywhere!

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Sections 1.3

Recall in Section 0.2–3 we agreed $\frac{dy}{dx} = f(x, y)$ tends to be harder than $\frac{dy}{dx} = f(x)$, That doesn't mean they are impossible.

> **Seperable Equation.** An first order ODE is *separable* if it can be written as $y' = f(x)g(y)$, where f and g are functions

Separable equations can be solved with Integration!

1.3 Separable equations

How can we manipulate $\frac{dy}{dx} = f(x)g(y)$ to solve the ODE?

Do you want to just multiply dx by both sides? What does that even mean?

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

$$y = h(x) \quad dy = h'(x)dx$$

$$\frac{1}{g(h(x))}h'(x) = f(x) \Rightarrow \int \frac{1}{g(h(x))} h'(x)dx = \int f(x)dx$$

$$\boxed{\Rightarrow \int \frac{1}{g(y)} dy = \int f(x) dy}$$

Despite the wondrous power of separable equations, there is still one minor issue. What happens when we can integrate, but we can't solve for y in a reasonable way?

> **Implicit Solutions.** A solution to an ODE not of the explicit form $y = h(x)$.

Example 1. Solve $(1 + x)dy - ydx = 0$. $\Rightarrow (1+x)dy = ydx \Rightarrow \frac{1}{y}dy = \frac{1}{1+x}dx$

$$\Rightarrow \ln|y| = \ln|1+x| + C$$

$$y = C_0(1+x)$$

We may not want to, but we can actually solve for y for this solution. Let's do that.

$$\frac{1}{\cot y}\,dy = \frac{x}{\sec x}\,dx \Rightarrow \int \tan y\,dy = \int x\cos x\,dx$$

$$\ln|\sec y| = x\sin x + \cos x + C$$

$$\sec y = C_0 \exp(x\sin x + \cos x)$$

Example 2. Solve $\sec(x)dy = x\cot(y)dx$

Example 3. You've found a dead body! Its temperature is 88.6° F at 2am and 78.6° F at 3am. The ambient air temperature is 68.6° F from midnight to 3am. Estimate the time of death.

$$\frac{dT}{dt} = k(T - T_{air})$$

Newton's Law of Cooling

T temp (°F)

t time (h)

Find murder o' Clock

$T(0) = 98.6$ cooling began

$$\frac{dT}{dt} = k(T - 68.6) \to \frac{1}{T-68.6}\,dT = \overset{k>0}{-k}\,dt$$

$$\ln|T-68.6| = -kt + C$$

$$\to T - 68.6 = C_0 e^{-kt}$$

$$T = C_0 e^{-kt} + 68.6$$

$$T(0) = C_0 + 68.6 \to C_0 = 30$$

$$T(t) = 30e^{-kt} + 68.6$$

See next page for Sol'n $\longrightarrow$

$$88.6 = T(t_0) = 68.6 + 30e^{-kt_0}$$

$$78.6 = T(t_0+1) = 68.6 + 30e^{-k(t_0+1)}$$

$$\frac{2}{3} = e^{-kt_0} \to \ln\frac{2}{3} = -kt_0$$

$$\to k = \frac{-1}{t_0}\ln\frac{2}{3}$$

$$\frac{1}{3} = e^{-k(t_0+1)} = e^{\frac{1}{t_0}\ln(\frac{2}{3})(t_0+1)}$$

$$= e^{\frac{1}{t_0}(t_0\ln\frac{2}{3} + \ln\frac{2}{3})}$$

$$= e^{\ln(\frac{2}{3}) + \frac{1}{t_0}\ln\frac{2}{3}}$$

$$= \frac{2}{3}e^{\frac{1}{t_0}\ln\frac{2}{3}}$$

$$\to \frac{1}{2} = e^{\frac{1}{t_0}\ln\frac{2}{3}}$$

$$\to \ln\frac{1}{2} = \frac{1}{t_0}\ln\frac{2}{3} \to t = \frac{\ln 2/3}{\ln 1/2} \approx 0.5849\text{ h} \sim 35\text{min}$$

$$\boxed{\text{MURDER O'CLOCK} \quad 1:25\text{ AM}}$$

T = body tempurature (°F)

t = time (h)

Newton's Law of Cooling: $\frac{dT}{dt} = k(T - T_{air})$

$$\rightsquigarrow \frac{dT}{T-68.6} = k\,dt \Rightarrow \ln|T-68.6| = kt + c$$

$$\Rightarrow T - 68.6 = C_0 e^{kt}$$

$$\Rightarrow T = C_0 e^{kt} + 68.6$$

Let $t=0$ be the time the body started cooling (98.6°F)

$$\Rightarrow 98.6 = C_0 e^{k(0)} + 68.6 \Rightarrow C_0 = 30$$

We have a system of 2 unknowns and 2 variables

$$\begin{cases} 88.6 = 30 e^{kt} + 68.6 \longrightarrow \frac{2}{3} = e^{kt} \\ 78.6 = 30 e^{k(t+1)} + 68.6 \longrightarrow \frac{1}{3} = e^{k(t+1)} = e^{kt} e^{k} \end{cases}$$

$$\frac{1}{3} = \frac{2}{3} e^{k} \rightarrow e^{k} = \frac{1}{2} \rightarrow k = \ln\left(\frac{1}{2}\right)$$

$$\longrightarrow \frac{2}{3} = e^{(\ln\frac{1}{2})t} \rightarrow \frac{2}{3} = \left(\frac{1}{2}\right)^{t} \rightarrow t = \frac{\ln 2/3}{\ln 1/2} = 0.5849\text{ h} \sim 35\text{ min}^{*}$$

$*$ = time after start of cooling

$$\Longrightarrow \text{Murder O'Clock} = 2\text{am} - 35\text{m} = \boxed{1{:}25\text{ am}}$$

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Sections 1.4

Recall in the last section we looked at some *"easy"* cases of $y' = f(x, y)$. Here's a slightly less easy one.

First Order Linear Equation. An ODE of the form

$$\frac{dy}{dx} + P(x)y = f(x)$$

is called *first order linear*. Additionally, we call this standard form for the first order linear equation.

1.4 First Order Linear Equations

How can we solve $\frac{dy}{dx} + P(x)y = f(x)$?

$$q(x)\frac{dy}{dx} + p(x)y + g(x) = 0$$

1) Divide by $q(x)$

$$\to \frac{dy}{dx} + \frac{p(x)}{q(x)}y = \frac{-g(x)}{q(x)}$$

$$\boxed{\to \underline{\frac{dy}{dx} + P(x)y} = f(x)}$$

standard form

Looks vaguely like a product rule

There are 5 easy steps!

1. Write in Standard Form
2. Find integrating factor

$$r(x) = \mu(x) = e^{\int P(x)dx}$$

3. Multiply both sides of standard form by μ

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

$$\frac{d}{dx}[\mu(x)y] = \mu'(x)y + \mu(x)y'$$

$$= P(x)\mu(x)y + \mu(x)\frac{dy}{dx}$$

4. Undo product rule

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x)$$

5. Integrate!

$$\int \frac{d}{dx}[\mu(x)y]\,dx = \int \mu(x)f(x)$$

$$\parallel$$

$$\mu(x)y \quad // \quad \rightsquigarrow \quad y = \frac{\int \mu(x)f(x)}{\mu(x)}$$

Let's look at an example!

Example 1. Find a general solution and find an interval on which the solution is defined.

$$\frac{dy}{dx} = y + e^x \quad \Leftrightarrow \quad y' - y = e^x$$

$$\mu(x) = e^{\int P(x)dx} = e^{\int -dx} = e^{-x}$$

$$\to \quad e^{-x}y' - e^{-x}y = \cancel{e^{-x}e^{x}} \; 1$$

$$\frac{d}{dx}\left[e^{-x}y\right] = 1 \to e^{-x}y = x + c \to \boxed{y = xe^x + ce^x}$$

Example 2. Solve $x\,dy = (x\sin(x) - y)\,dx$

$$x\frac{dy}{dx} = x\sin x - y \to xy' + y = x\sin x \to y' + \frac{y}{x} = \sin x \qquad (x \neq 0)$$

$$\mu = e^{-\int \frac{1}{x}dx} = e^{\ln|x|} = x \rightsquigarrow xy' + y = x\sin x \to \frac{d}{dx}\left[xy\right] = x\sin x$$

$$\to xy = \int x\sin x\,dx \qquad \begin{array}{cc} +x & \sin x \\ -1 & -\cos x \\ 0 & -\sin x \end{array} \qquad \to xy = -x\cos x + \sin x + c$$

$$\to \boxed{y = -\cos x + \frac{\sin x}{x} + c\cdot\frac{1}{x}}$$

Example 3. Solve $y' = 2y + x(e^{3x} - e^{2x})$ given the initial condition of $y(0) = 2$.

$$y' - 2y = xe^{3x} - xe^{2x} \qquad \mu(x) = e^{-2x}$$

$$2 = -1 + c$$

$$\to c = 3$$

plug back in

$$\frac{d}{dx}\left[e^{-2x}y\right] = xe^x - x \to e^{-2x}y = xe^x - e^x - \frac{1}{2}x^2 + c$$

$$\begin{array}{cc} +x & e^x \\ -1 & e^x \\ +0 & e^x \end{array}$$

IVP

$$\to \boxed{y = -\frac{1}{2}x^2e^{2x} + xe^{3x} - e^{3x} + 3e^{2x}}$$

Example 4. Initially, 50 pounds is dissolved in a large tank holding **300 gallons of water**. A brine solution is pumped into the tank at a rate of **3 gallons per minute**, and the well-stirred solution is then pumped out <u>at the same rate</u>. If the concentration of the solution entering is **2 pounds per gallon**, determine the amount of salt in the tank at time t.

How much salt is present after 50 minutes? After a long time?

$A(t)$ amount of salt @ time t (lbs)

t time (min)

$$\frac{dA}{dt} = \left(\text{rate in} - \text{rate out}\right) = \left(\frac{3\text{ gal}}{\text{min}}\cdot\frac{2\text{ lbs}}{\text{gal}} - \frac{3\text{ gal}}{\text{min}}\cdot\frac{A(t)}{300}\frac{\text{lbs}}{\text{gal}}\right) = 6 - \frac{A(t)}{100}$$

$$A' + \frac{1}{100}A = 6$$

$$\mu = e^{0.01t} \rightsquigarrow A = 600 + \frac{Ce^{-t}}{100} \to C = -550$$

Begin 12 Sept

Review: - Seperable $\frac{dy}{dx} = f(x)g(y)$

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

- Linear $\frac{dy}{dx} + P(x)y = f(x)$

Let $\mu(x) = e^{\int P(x)\,dx}$

then $y = \frac{1}{\mu(x)} \int \mu(x) f(x)\,dx$

"The Day of Weird Subs"

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Sections 1.5

We have learned some really neat tricks to leverage separability and linearity and solve ODEs. When all of those things fail, here's the next thing you try:

> **Homogeneous ODE**. A first order ODE is called homogeneous if it can be written as
>
> $$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

You may notice that this word has been used before. We give a different definition here because it made sense to someone at some point. Use context to determine which version of "homogeneous" you're dealing with.

1.5 Substitution

How can we solve $xy' + y + x = 0$ with initial condition $y(1) = 1$?

Subtract x, divide by it!

$\rightarrow y' + \frac{1}{x} y = -1$ ✱

We could use linear techniques ... or substitute

$V = \frac{Y}{x} \Rightarrow Y = xV$

$\hookrightarrow y' = V + xV'$

$V + xV' + V = -1$

$V' + \frac{2}{x} V = \frac{-1}{x}$

this eqn is also linear ... so we got nowhere. Do Integrating Factor method on either ✱

$\Rightarrow Y = \frac{3 - x^2}{2x}$ (use IVP to solve for C)

Y is our dependent variable, so get rid of them all w/ v

Substitution problems are a lot like ice cream. They come in many flavors, and if you have too many, your brain freezes.

Example 1. Solve the IVP $2yy' + 1 = y^2 + x$, $y(0) = 1$.

Not linear!

$V = y^2 \rightarrow V' = 2yy'$

Some times substitutions help

$V' + 1 = V + x$

Linear!

$\Rightarrow V' - V = x - 1$

$\mu(x) = e^{\int -dx} = e^{-x}$

$+ x \quad e^{-x}$

$- 1 \quad -e^{-x}$

$+ 0 \quad e^{-x}$

$\rightarrow V = \frac{1}{e^{-x}} \int (xe^{-x} - e^{-x}) dx$

$= \frac{1}{\cancel{e^{-x}}} \left(-x\cancel{e^{-x}} \cancel{- e^{-x}} \cancel{+ e^{-x}} + c \right)$

$= Ce^{x} - x$

$y^2 = Ce^{x} - x$

$\Rightarrow 1 = Ce^{0} - 0$

$\Rightarrow y^2 = e^{x} - x$

You may find it helpful to know the contents of this chart:

If you see...	Try this substitution!
xy'	$v = \frac{y}{x}$
yy'	$v = y^2$
y^2y'	$v = y^3$
$(\cos y)y'$	$v = \sin y$
$(\sin y)y'$	$v = \cos y$
$y'e^y$	$v = e^y$

Example 2. Bernoulli's Equation!

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \quad n \in \mathbb{R}$$

sub: $v = y^{\boxed{1-n}}$ NOT $n-1$, easy mistake

$v' = (1-n)\,y^{-n}\,y'$

divide by y^n

$y' + P(x)y = f(x)y^n$

$y^{-n}y' + P(x)y^{1-n} = f(x)$

$\frac{1}{1-n}v' + P(x)v = f(x)$

$v' + (1-n)P(x)v = f(x)(1-n)$ Linear!

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Sections 1.6

Idea: we can qualitatively study autonomous eqns w/o solving them!

Recall,

Autonomous Equations. First order autonomous ODEs are of the form

$$\frac{dx}{dt} = f(x)$$

← No indep variables as function input!

Also recall,

Newton's Law of Cooling.

$$\frac{dx}{dt} = -k(x - A)$$

Autonomous

Note that $x = A$ is a constant solution to any Newton's Law of Cooling problem.

1.6 Autonomous Equations

Constant solutions for an ODE are called equilibrium solutions (or equilibria solutions if you have more than one).

Any point x_0 on the x-axis where $\frac{dx}{dt} = f(x_0) = 0$ is called a critical point. Why?

derivative is zero! See Calc. I

Stability of Equilibria.

An equilibrium is *stable* (or attracting) if nearby solutions approach it as $t \to \infty$.
unstable (or repelling) if nearby solutions move away from it as $t \to \infty$.

Equilibria that are not stable or unstable are called *shunt* (or indifferent).

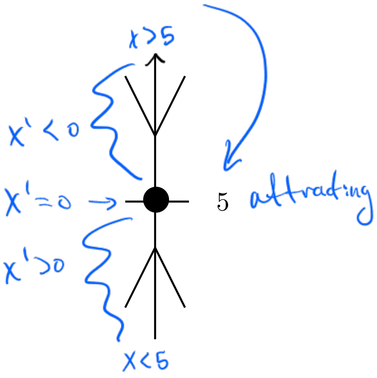
Goal: understand behavior of autonomous eqns through the study of critical / equilibrium points

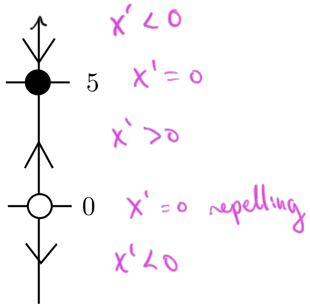
Think first derivative test

Compare the phase diagrams or phase portraits of the following ODEs equilibria.

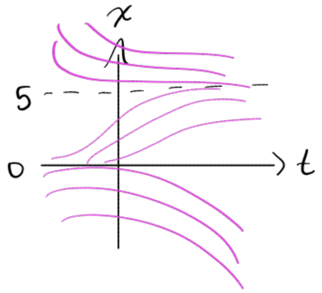
$$x' = -0.3(x-5) \quad \text{and} \quad x' = 0.1x(5-x)$$

$x'=0$ @ $x=5$ $x'=0$ @ 0 and 5





Logistic map



How do we construct these phase diagrams?

1.

2.

3.

4.

Example 1. Logistic growth with harvesting:

$$\underbrace{\frac{dx}{dt} = kx(M-x)}_{\text{general logistic growth}} - h \text{ where } k=1 \text{ and } M=2$$

↳ $x' = x(2-x) - h$

$= -x^2 + 2x - h$

$\frac{-2 \pm \sqrt{4-4h}}{-2}$

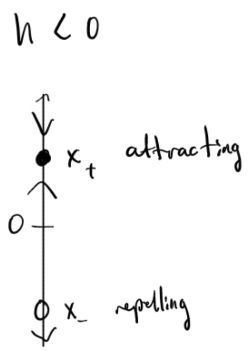
K = growth constant

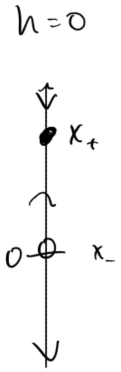
M = carrying capacity

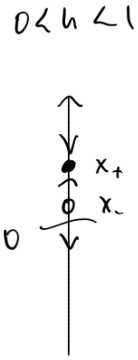
h = harvesting parameter

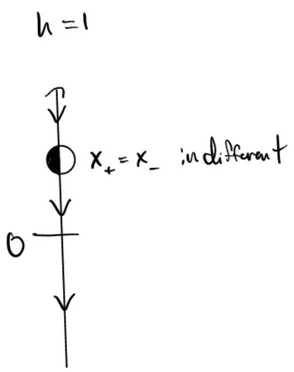
→ $x_\pm = 1 \pm \sqrt{1-h}$

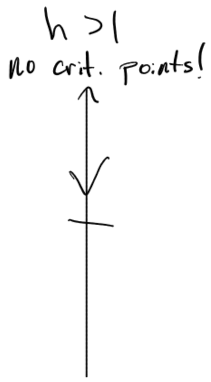
Bifurcation theory!











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Section 1.7

Sometimes, we can't find a solution. If I just pick an ODE out of a bag, it is not going to be solved through any of the techniques we've looked at so far. So what can we do?

1.7 Euler's Method

Euler's Method is a way to approximate $x(t_1), x(t_2), x(t_3), ..$ where $t_0 < t_1 < t_2 < ....$

We accomplish this through the definite integral of both sides of

$$\begin{aligned} x' &= f(t,x) \\ x(t_1) - x(t_0) &= \int_{t_0}^{t_1} f(t,x(t))dt \end{aligned}$$

Integrate, FTC says...

This implies that

$$x(t_1) = x_0 + \boxed{\int_{t_0}^{t_1} f(t,x(t))dt.}$$

Almost certainly impossible to directly compute

We can use your favorite Riemann Sum evaluation technique. We'll use the Left Hand Rule.

What is $x(t_1)$? It's our first approximation; let's call it x_1.

$$x_1 = x_0 + \underbrace{(t_1 - t_0)}_{\text{step } s} f(t_0, x(t_0))$$

We tend to make our t_i's evenly spaced apart to create consitent step size s.

How can we approximate $x(t_2)$ (which we call x_w)?

$$x_2 = x_1 + sf(t_1, x_1)$$

How can we approximate $x(t_n)$ (that is, x_n)?

$$\leadsto x_{n+1} = x_n + sf(t_n, x_n)$$

What do we need to consider when determine how many steps to take in our Euler Method approximation?

Let's look at an example!

Example 1. $x' = x, x(0) = 1$. Given a step size of 0.2 and $t_0 < t < 1$.

↳ general soln: $x(t) = e^t$

$x(0) = 1$

$x(1) = e$

$t=0$ $x_0 = 1$

$t=0.2$ $x_1 = 1 + 0.2\overset{x}{(1)} = 1.2$

$t=0.4$ $x_2 = 1.2 + 0.2(1.2) = 1.44$

$t=0.6$ $x_3 = 1.44 + 0.2(1.44) = 1.728$

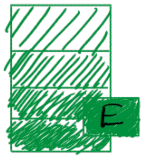
$t=0.8$ $x_4 = 1.728 + 0.2(1.728) = 2.0736$

$t=1$ $x_5 = 2.0736 + 0.2(2.0736) = 2.48832$

For a more in depth analysis of step size, see page 24 of Lebl.

Example 2. Computer Time

Excel!



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Section 1.8

1.8 Exact Equations

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$, so we could graph graph f in $\mathbb{R}^3$ by $z = f(x, y)$. We could also take the *total differential* of f as follows:

$$\begin{aligned} z &= f(x,y) \\ dz &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \end{aligned}$$

For example, if $f(x, y) = x^2 + y^2$, then

$$dz = 2x dx + 2y dy$$

Thus, $2x\,dx + 2y\,dy = 0$ has

$$x^2+y^2 = C$$

as the general solution.

Exact Equations. The differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is an *exact differential equation* if the left hand side of the equation is an exact differential.

In other words, $M(x, y)\,dx + N(x, y)\,dy = 0$ is an exact differential equation if there is some function $f\colon \mathbb{R}^2 \to \mathbb{R}$, often called a *potential function*, such that

$$df = M(x, y)\,dx + N(x, y)\,dy.$$

Criterion for Exactness. Let $M(x, y)$ and $N(x, y)$ be continuous with continuous partial derivatives in some rectangular region R in $\mathbb{R}^2$. Then $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Why? Note that $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact if and only if there is a function f such that

Then by Clairut's Theorem, this is true if and only if

↳ Symmetry of partial derivatives

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Example 1. Is $\dfrac{dy}{dx} = \dfrac{-2x-y}{x-1}$ exact?

$$(x-1)\,dy = -(2x+y)\,dx$$

$$(2x+y)\,dx + (x-1)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1 \qquad \checkmark \text{ Exact!}$$

Example 2. Solve $2xy\,dx + (x^2-1)\,dy = 0$

$$\text{Exact!} \quad \frac{\partial M}{\partial y} = 2x \overset{\checkmark}{=} 2x = \frac{\partial N}{\partial x}$$

Guess of potential fcn:

$$M = \frac{\partial f}{\partial x} \rightarrow f = \int M\,dx = \int 2xy\,dx = x^2y + C(y)$$

$$f = \int N\,dy = \int (x^2-1)\,dy = x^2y - y + C(x)$$

but also ←

$$\underline{x^2-1} = N = \frac{\partial f}{\partial y} = \underline{x^2 + c'(y)} \qquad \rightarrow f = x^2y \underline{-y + C_0}$$

$$\boxed{\text{So answer: } f(x,y) = x^2y - y = c}$$

$$\rightarrow c'(y) = -1 \rightarrow c(y) = -y + C_0$$

Example 3. Solve $\underbrace{(\sin(y) - y\sin(x))}_{N}\,dx + \underbrace{(\cos(x) + x\cos(y) - y)}_{N}\,dy = 0$

$$\frac{\partial M}{\partial y} = \cos y - \sin x \qquad \frac{\partial N}{\partial x} = -\sin x + \cos y \qquad \text{Exact}\checkmark$$

$$f = \int \frac{\partial f}{\partial x}\,dx = \qquad \text{complete later}$$

Example 4. Solve $(3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy = 0$

$$\frac{\partial M}{\partial y} = 3x^2 + e^y \qquad \frac{\partial N}{\partial x} = 3x^2 + e^y$$

$$f = \int \frac{\partial f}{\partial x}\,dx = \int M\,dx = \int (3x^2y + e^y)\,dx = x^3y + xe^y + C(y)$$

$$\boxed{f(x,y) = x^3y + xe^y - y^2 = c}$$

$$N = \frac{\partial \int \frac{\partial f}{\partial x}\,dx}{\partial y} = x^3 + xe^y + C'(y) \rightarrow C(y) = -y^2$$

Example 5. Solve $(3x\cos(3x) + \sin(3x) - 3)\,dx + (2y + 5)\,dy = 0$

It's a trap! Seperable

Example 6. Solve $(x + y)\,dx + (x\ln(x))\,dy = 0$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = \ln x + 1$$

Not exact

Linear in y

$$(x+y) + x\ln x \frac{dy}{dx} = 0$$

$$x \ln x \frac{dy}{dx} = -(x+y)$$

$$\frac{dy}{dx} = \frac{-(x+y)}{x\ln x} \Rightarrow \frac{dy}{dx} + \frac{1}{x\ln x}y = \frac{-1}{\ln x}$$

Example 7. Solve $y(x + y + 1)\,dx + (x + 2y)\,dy = 0$

$$\frac{\partial M}{\partial y} = x+y+1+y \qquad \frac{\partial N}{\partial x} = 1$$

Not exact

$$(x+2y)\frac{dy}{dx} = -yx - y^2 - y$$

Not linear!

$$v = xy \rightarrow y = \frac{x}{v} \rightarrow \frac{dy}{dx} = \frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx} \; . \; . \; .$$

Now it's exact

Not Exact

Example 8. Solve $(\overset{-2}{\cancel{-}}xy\sin(x) + 2y\cos(x))\,dx + (2x\cos(x))\,dy = 0$

$$\frac{\partial M}{\partial y} = -x\sin x + 2\cos x \qquad \frac{\partial N}{\partial x} = -2x\sin x + 2\cos x$$

$$F = \int \frac{\partial f}{\partial y}\,dy = \int N\,dy = \int 2x\cos x\,dy = 2xy\cos(x) + C(x)$$

$$\frac{\partial \int \frac{\partial f}{\partial y}\,dy}{\partial x} = 2y\cos x - 2xy\sin x + \overset{=0}{C'(x)}$$

$$\Longrightarrow \boxed{f(x,y) = 2xy\cos(x) = c}$$

Example 9. Solve $2xe^x - y + 6x^2 = \frac{x\,dy}{dx}$

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Section 2.1

2.1 Second Order ODEs

Second Order Linear ODEs A second order linear ODE is of the form

$$A(x)y'' + B(x)y' + C(x)y = D(x)$$

However, we can always make our lives easier and divide by $A(x)$ to achieve

$$y'' + p(x)y' + q(x)y = f(x)$$

General form

Superposition Theorem. If y_1 and y_2 are solutions to the second order linear homogenous equation $y'' + p(x)y' + q(x)y = 0$, then for any constants C_1, C_2,

$$y = C_1y_1 + C_2y_2$$

is also a solution.

Let's take another look at the Fundamental Theorem for Existence and Uniqueness!

Fundamental Theorem for Existence and Uniqueness (revisited). Suppose p, q, and f are continuous on some interval I and a, b_0, b_1 are constants such that $a \in I$. The ODE

must contain initial conditions

there's two of 'em!

$$y'' + p(x)y' + q(x)y = f(x)$$

has exactly one solution y on I satisfying $y(a) = b_0$ and $y'(a) = b_1$.

Picard, II

Only works for linear second order

Example 1. Verify $y = b_0 \cos(kx) + \frac{b_1}{k}\sin(kx)$ is a unique solution to to $y'' + k^2y = 0, y(0) = b_0, y'(0) = b_1$.

$$\underbrace{p(x) = 0 \quad q(x) = k^2 \quad f(x) = 0}_{\text{Continuous everywhere! on any interval! Bahahah!}}$$

$$\frac{dy}{dx} = -kb_0\sin(kx) + b_1\cos(kx)$$

$$\frac{d^2y}{dx^2} = -k^2b_0\cos(kx) - kb_1\sin(kx)$$

$$\frac{d^2y}{dx^2} + k^2y = \left(-k^2b_0\cos(kx) - kb_1\sin(kx)\right) + k^2\left(b_0\cos(kx) + \frac{b_1}{k}\sin(kx)\right) \overset{\checkmark}{\equiv} 0$$

$$y(0) = b_0\cos(0) + \frac{b_1}{k}\sin(0) = b_0 \checkmark \qquad y'(0) = -kb_0\sin(0) + b_1\sin(0) = b_1 \checkmark$$

What does it mean for a set of functions to be linearly dependent?

$y_1, \ldots, y_n : I \to \mathbb{R}$ are linearly dependent if $\exists\, c_1, \ldots, c_n \in \mathbb{R}$ (not all zero)

when $\sum_{k=1}^{\infty} c_k y_k = 0 \quad \forall x \in I$

Example 2. Show $\sinh(x)$ and $\cosh(x)$ are linearly independent. (Recall, $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.)

$$c_1 \sinh(x) + c_2 \cosh(x) = \frac{c_1}{2}\left(e^x - e^{-x}\right) + \frac{c_2}{2}\left(e^x + e^{-x}\right) = \frac{(c_1+c_2)}{2} e^x + \frac{(c_2-c_1)}{2} e^{-x}$$

Sps BWOC $\{\sinh x, \cosh x\}$ dependent. Then $\rightarrow = 0 \ \forall x \in \mathbb{R}$. Note $e^x \neq 0 \ \forall x \in \mathbb{R}$,

So $c_1 \sinh(x) + c_2 \cosh x = 0$ only if $\underbrace{\frac{c_1+c_2}{2} = 0 \text{ and } \frac{c_2-c_1}{2} = 0}$

$\hookrightarrow c_1 = c_2 = 0$ ↯ linearly independent!

> **Theorem.** Let p, q be continuous functions and y_1, y_2 solutions to the ODE
>
> $$y'' + p(x)y' + q(x)y = 0.$$
>
> linearly independent sol'ns
>
> Then $y = c_1y_1 + c_2y_2$ is the general solution to the ODE. for $c_1, c_2 \in \mathbb{R}$

Example 3. Find the general solution to $y'' + y = 0$

$$\left.\begin{matrix} y_1 = \sin x \\ y_2 = \cos x \end{matrix}\right\} \text{Both solns to } y'' + y = 0$$

Assume BWOC $c_1 \sin x + c_2 \cos x = 0$

$$\Rightarrow \frac{-c_1}{c_2} \frac{\sin x}{\cos x} = 1 \Rightarrow \frac{-c_1}{c_2} \tan x = 1$$

Not true for $x = 0$

$\frac{-c_1}{c_2} \tan x = 0$

Lemma: $\sin x$ and $\cos x$ are linearly independent.

What do we do when we already have one solution?

y_1 is a soln to $y'' + p(x)y' + q(x)y = 0$

$y_2 = v(x)\, y_1(x)$ for some $v(x)$

$y_2' = v'y_1 + vy_1'$

$y_2'' = v''y_1 + 2v'y_1' + vy_1''$

$$v''y_1 + 2v'y_1' + vy_1'' + p(y)(v'y_1 + vy_1') + q(x)vy_1 = 0$$

$$v\underbrace{(\cancel{v}y_1'' + p(x)\cancel{v}y_1' + q(x)\cancel{v}y_1)}_{0} + v''y_1 + 2v'y_1' + p(x)v'y_1 = 0$$

$$y_1v'' + (2y_1' + p(x)y_1)v' = 0$$

$w = v'$

$$y_1w' + (2y_1' + p(x)y_1)w = 0$$

$$w' + \left(\frac{2y_1'}{y_1} + p(x)\right)w = 0$$

First order linear!

$$\mu = e^{\int\left(\frac{2y_1'}{y_1} + p(x)\right)dx} = e^{2\ln|y_1| + \int p(x)dx} = y_1^2 e^{\int p(x)dx}$$

$$\Rightarrow \frac{d}{dx}[\mu w] = 0$$

$\mu w = C$

$y_1^2 e^{\int p(x)dx} w = C$

$$v' = w = \frac{C}{y_1^2 e^{\int p(x)dx}} = Ce^{-\int p(x)dx} y_1^{-2}$$

let $c = 1$ (only need one sol'n)

$$v = \int e^{-\int p(x)dx} y_1^{-2}\, dx$$

$$\boxed{\Rightarrow y_2 = y_1 \int \frac{e^{-\int p(t)dt}}{y_1^2} dx}$$

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Sections 2.2–3

2.2 Constant Coefficient Second Order Linear ODEss

Constant Coefficient Second Order Linear ODEs A second order linear ODE is of the form

$$ay'' + by' + cy = f(x)$$

However, for right now we are going to focus on the much easier to solve:

$$ay'' + by' + cy = 0 \qquad a, b, c \in \mathbb{R}$$

Let's *guess* a solution of $y = e^{rx}$. What does this achieve?

Auxillary equs

$$e^{rx}\left(\boxed{ar^2 + br + c}\right) = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Recall from prior courses,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What can our roots look like?

$\mathbb{C}$

We have a strategy to find solutions based on the form our roots take.

- **2 Real Roots:**

$r_1 \neq r_2$

$y_1 = e^{r_1 x}, \; y_2 = e^{r_2 x}$ ← linearly indep.!

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

1 rep. ℝ root → discriminant = 0

$\Rightarrow r = \frac{-b}{2a}$

- **1 Real Root:**

$$Y_1 = e^{rx} \Rightarrow Y_2 = Y_1 \int \frac{e^{-\int p(t)\,dt}}{Y_1^2}\,dx = e^{rx} \int \frac{e^{-\int \frac{b}{a}dx}}{e^{2rx}}\,dx = e^{rx} \int \frac{e^{-\frac{b}{a}x}}{e^{2rx}}\,dx = e^{rx} \int \frac{e^{-\frac{b}{a}}}{e^{-2\frac{b}{2a}}}\,dx = e^{rx} \int 1\,dx \boxed{= xe^{rx}}$$

$$\rightsquigarrow \boxed{Y = C_1 e^{rx} + C_2 x e^{rx}}$$

- **2 Complex Roots:** ℂ my beloved

$r_1 = \alpha + i\beta$

$r_2 = \alpha - i\beta$

Euler: $e^{i\theta} = \cos\theta + i\sin\theta$

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x) \longleftarrow$$

$$Y_2 = \text{---}\,''\,\text{---}\,''\,\text{---} = e^{\alpha x}(\cos\beta x - i\sin\beta x) \longleftarrow$$

ℂ sol'ns!

$$\boxed{Y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x}$$

$$Y_3 = \frac{1}{2}(Y_1 + Y_2); \quad Y_4 = \frac{1}{2}(Y_1 - Y_2)$$

$$= e^{\alpha x}\cos\beta x \quad ; \quad = e^{\alpha x}\sin\beta x \longleftarrow$$

ℝ sol'ns !

Example 1. Find the general solution for

$$y^{(4)} + y^{(3)} - 3y'' - 2y' = 0.$$

Aux eqn

Guess $y = e^{rt}$

$$e^{rt}(r^4 + r^3 - 3r^2 - 2r) = 0$$

↳ $r = 0$ trivial soln

It's golden!

$$e^{rt}(r)(r+2)(r-\varphi)(r-\bar{\varphi})$$

$$y = \overset{r=0}{C_1} + \overset{r=-2}{C_2 e^{-2t}} + \overset{r=\varphi}{C_3 e^{\varphi t}} + \overset{r=\bar{\varphi}}{C_4 e^{\bar{\varphi} t}}$$

Idea: Higher order constant coeff. linear diff eqs can be solved using $y = e^{rt}$ and factoring the aux eqn into products of quadratic roots!

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Sections 2.5

2.5 Nonhomogeneous Equations

Constant Coefficient Linear Nonhomogeneous ODEs. A linear nonhomogeneous ODE with constant coefficients is of the form

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x),$$

where $f(x) \neq 0$.

Lebl calls the LHS of this equation Ly, where L is a linear transformation. That is,

$L: C \to C$ via $L(y) = \sum_{k=0}^{n} a_k y^{(k)}$

set of continuous fxns

$$Ly = a_n y^{(n)} + \cdots + a_1 y' + a_0 y,$$

where L is the function that turns a function y into this very specific linear combination of y and its derivatives. Can you show that L is a linear transformation?

To solve a nonhomogeneous equation, first solve the *associated homogeneous equation*,

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0, \quad \Longleftrightarrow \quad Ly = 0 \quad \text{or} \quad L[y] = 0$$

and call the general solution y_c (c for "complementary"). That's right. Just pretend that $f(x)$ was never there.

Next, find a particular solution for the original nonhomogeneous equation (drat! $f(x)$ has returned!)

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = \bigcirc\!\!\!\!\!\! f(x), \quad \Longleftrightarrow \quad Ly = f(x) \quad \text{or} \quad L[y] = f(x)$$

Sometimes called the forcing fxn

and call it y_p (p for "particular").

Theorem. The general solution to the nonhomogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x)$$

is

$$y = y_c + y_p,$$

where y_c is the general solution to the associated homogeneous equation, and y_p is *any* particular solution to the original nonhomogeneous equation.

Proof of this theorem follows from the linearity of L.

One question remains: How do we get that one particular solution we need? Yep. That is the hard part. We'll study two methods:

1. The Method of Undetermined Coefficients[1] Algebra intensive
2. Variation of Parameters[2] Calculus intensive

---

[1] This is glorified guess and check.

[2] This is often called "Var of Parm," which definitely sounds more delicious.

2.5.1 The Method of Undetermined Coefficients

Example 1. Find the general solution for

$$y'' - 4y' - 12y = \sin 2t.$$

$$e^{rt}(r^2 - 4r - 12) = \sin 2t$$

$$\rightsquigarrow y_h = y_c = C_1 e^{6t} + C_2 e^{-2t}$$

Guess $y_p = A \sin 2t + B \cos 2t$

the undetermined coefficient(s)

$$y_p' = 2A\cos 2t - 2B\sin 2t$$

$$y_p'' = -4A\sin 2t - 4B\cos 2t$$

$$-4A\sin 2t - 4B\cos 2t - 8A\cos 2t + 8B\sin 2t$$

$$-12A\sin 2t - 12B\cos 2t$$

$$\sin 2t = \underbrace{(-4A + 8B - 12A)}_{\substack{-16A+8B \\ \shortparallel \\ 1}}\sin 2t + \underbrace{(-4B - 8A - 12B)}_{\substack{-8A-16B \\ \shortparallel \\ 0}}\cos 2t$$

$$\Rightarrow A = \frac{-1}{20},\ B = \frac{1}{40}$$

$$\rightsquigarrow \boxed{y = y_h + y_p = C_1 e^{6t} + C_2 e^{-2t} + \frac{-1}{20}\sin 2t + \frac{1}{40}\cos 2t}$$

Example 2. Find the general solution for

$$y'' - 4y' - 12y = 2t^3 - t + 3.$$

$$y_h = C_1 e^{6t} + C_2 e^{-2t}$$

again!

$$\left.\begin{aligned} y_p &= At^3 + Bt^2 + Ct + D \\ y_p' &= 3At^2 + 2Bt + C \\ y_p'' &= 6At + 2B \end{aligned}\right\}$$

$$A = \frac{-1}{6},\ B = \frac{1}{6},\ C = \frac{-1}{9},\ D = \frac{-5}{27}$$

$$\boxed{y = C_1 e^{6t} + C_2 e^{-2t} - \frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}}$$

Example 3. Find the general solution for

$$y'' - 4y' - 12y = te^{4t}.$$

Superposition Revisited.[a] Let y_1 be a solution for

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f_1(x),$$

and y_2 be a solution for

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f_2(x),$$

Then for any constants k_1 and k_2, $k_1 y_2 + k_2 y_2$ is a solution for

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = k_1 f_1(x) + k_2 f_2(x).$$

---

[a] Now with more super-ness!

2.5.2 Variation of Parameters

Here's a fun thing:

The Wronskian. Let y_1 and y_2 be continuous on some interval I. Then the *Wronskian* of y_1 and y_2, denoted by $W(y_1, y_2)$, is given by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Theorem: This is not the Wronskian you're looking for.
Let y_1 and y_2 be continuous on some interval I. Then $W(y_1, y_2) = 0$ for all $x \in I$ if and only if y_1 and y_2 are linearly dependent on I.

Example 4. Show that $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are linearly independent if and only if $r_1 \neq r_2$.

$$W(y_1, y_2) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = r_2 e^{(r_1+r_2)t} - r_1 e^{(r_1+r_2)t} = (r_2 - r_1) e^{(r_1+r_2)t}$$

$$\neq 0 \ \forall t \in \mathbb{R}$$

iff $r_1 \neq r_2$

Var of parm is great if you have a second order nonautonomous, nonhomogeneous equation and you really like integrals. Suppose first that you have

$$y_c = c_1 y_1 + c_2 y_2,$$

the complementary solution for the second order nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Note that we've normalized our equation so that there is no coefficient on y''. The big advantage of Var of Parm is that you *don't have to have constant coefficients.* Indeed, p and q can be any gross function of x you want.

Since we have y_c, all we need is y_p, so let's guess $y_p = u_1(x) y_1 + u_2(x) y_2$

where u_1 and u_2 are nonconstant functions of x. This looks gross, so we'll suppress all the (x)'s and have

$$y_p'$$

To get started, we need derivatives of y_p. Well,

which is, again, gross. Now we're gonna make an assumption that may seem like a total scam. This is fine. I promise it will be fine…eventually. For now, though, let's just assume

With this assumption, we now have

Plugging this in to our original nonhomogeneous equation, we have

After some algebra, we have

which is pretty great. Now we can combine this with the assumption we made earlier. It turns out that making this assumption *does* eliminate some of the possible solutions. Do we care? Not really. We only need one y_p! Now we have two equations:

Solving the first equation for u_1, we have

Substituting this into the second equation, we have

or, after algebra,

We could do some similar algebra to solve for u_1. Ultimately, we end up with

This gives us the following fun theorem:

Var of Parm. For the ODE

$$y'' + p(x)y' + q(x)y = f(x)$$

with complementary solution $y_c = c_1y_1 + c_2y_2$, a particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 f(x)}{W(y_1, y_2)}\,dx + y_2 \int \frac{y_1 f(x)}{W(y_1, y_2)}\,dx$$

Example 5. Find the general solution for TYPO:

$$y'' - 2y' \underset{+y}{} = \frac{e^x}{x^2+1}.$$

$r^2 - 2r + 1$

$(r-1)^2$

$W(y_1, y_2) = e^{2t}$

$Y_h = c_1e^t + c_2te^t$

$$Y_p = -e^t \int \frac{t\cancel{e^t} \cdot \frac{\cancel{e^t}}{t^2+1}}{\cancel{e^{2t}}}\,dt + te^t \int \frac{\cancel{e^t}\,\frac{\cancel{e^t}}{t^2+1}}{\cancel{e^{2t}}}\,dt$$

$$= -e^t \int \frac{t}{t^2+1}\,dt + te^t \int \frac{1}{t^2+1}\,dt = -\frac{1}{2}e^t \ln(1+t^2) + te^t \arctan t$$

$Y = C_1e^t + C_2te^t + ☆$

Example 6. Find the general solution for

$$2y'' + 18y = 6\tan 3x.$$

Example 7. Find the general solution for

$$xy'' - (x+1)y' + y = x^2.$$

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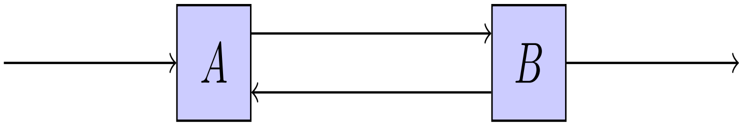
Section 3.1

3.1 Systems

Modeling pathogens, travel, etc



x is the amount in A
y is the amount in B

$$\frac{dx}{dt} = f(x, y, t) \quad \text{and} \quad \frac{dy}{dt} = g(x, y, t)$$

Example 1. $x' = x$ and $y' = x - y$

First order system in two variables → Soln is two fxns

One of these is significantly easier than the other.

$$\boxed{x = c_1 e^t} \rightsquigarrow y' = c_1 e^t - y \rightsquigarrow y' + y = c_1 e^t$$

$$\mu = e^x \to \frac{d}{dt}[ye^t] = c_1 e^{2t} \to ye^t = \frac{c_1}{2} e^{2t} + c_2$$

$$\hookrightarrow \boxed{y = \tfrac{1}{2} c_1 e^t + c_2 e^{-t}}$$

Example 2. $x' = 2y - x$ and $y' = x$

$y'' = x'$

$$x' = \underline{y'' = 2y - y'} \rightsquigarrow y'' + y' - 2y = 0 \rightsquigarrow r^2 + r - 2$$

$$(r+2)(r-1)$$

$$r = -2, +1$$

$$\boxed{y = c_1 e^{-2t} + c_2 e^t \rightsquigarrow x = y' = -2c_1 e^{-2t} + c_2 e^t}$$

Example 3. Turn the third order equation, $y''' = 2y'' - t^2 y' + \cos(t)y$, into a system of first order equations.

$$z = 2w - t^2 x + \cos(t) y$$

Turn any nth order linear into first order system

$$x = y'$$

$$w = x' = y''$$

$$z = w' = x'' = y'''$$

1st order system of 3 variables

$$\begin{cases} x' = w \\ y' = x \\ w' = 2w - t^2 x + \cos t \; y \end{cases}$$

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Section 3.3

3.3 Linear Systems

Given a n^{th} order linear or linear system of n equations in n variables,

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + \ldots + a_{1n}(t)x_n + f_1(t) \\ \vdots &\quad \vdots \quad\quad \ddots \quad \vdots \quad\quad \vdots \\ x_n' &= a_{n1}(t)x_1 + \ldots + a_{nn}(t)x_n + f_n(t) \end{aligned}$$

Let's write it in matrix equation form:

$$\overset{\text{derivative vector}}{\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}} = \overset{\text{Coeffs matrix}}{\begin{bmatrix} a_{11}(t) & \ldots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \ldots & a_{nn}(t) \end{bmatrix}} \overset{\text{frn vector}}{\begin{bmatrix} x_1 \\ \ddots \\ x_n \end{bmatrix}} + \overset{\text{non-homog ''forcing'' vector}}{\begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}}$$

This gives us

$$✱ \quad \vec{X}' = A\vec{x} + \vec{f}$$

supress the 't's!

Everything depends on t, so ignore it

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$$

$$\frac{d}{dt}\left[\vec{X}\right] = \frac{d}{dt}\begin{bmatrix} X_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

$$\int \vec{X} \overset{\text{def}}{=} \begin{bmatrix} \int x_1(t) \\ \vdots \\ \int x_n(t) \end{bmatrix}$$

What are some key things to keep in mind about this?

- Solutions to ✱ are vectors of frns! $\vec{X} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$
-

> **Superposition Revisited** If **x'** $= A\mathbf{x}$ is an $n \times n$ homogenous system, then any linear combination of solutions is a solution. Moreover, if $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linear independent, then
>
> $$\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$
>
> is the general solution.

Note

$$\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n = \underbrace{\overset{\text{Vectors of frns}}{\left[\vec{\mathbf{x}}_1(t)\ldots\vec{\mathbf{x}}_n(t)\right]}}_{\text{Matrix}} \overset{\vec{c}}{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}} = \mathbb{X}(t)\vec{\mathbf{c}}$$

It's fundamental!

We call $\mathbb{X}(t)$ the fundamental matrix (solution). It is a matrix whose columns are n linearly independent solutions to the system.

Example 1. Given $x' = -2x + 2y$ and $y' = 2x - 5y$. Build the fundamental matrix $\mathbb{X}$.

$$\vec{\eta}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2x+2y \\ 2x-5y \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix}\vec{\eta}$$

Suppose we're told

$$\vec{\eta}_1 = e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{\eta}_2 = e^{-6t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

LHS $\vec{\eta}_1' = \begin{bmatrix} -2e^{-t} \\ -e^{-t} \end{bmatrix}$

RHS $\begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix}\vec{\eta}_1 = \begin{bmatrix} -4e^{-t}+2e^{-t} \\ 4e^{-t}-5e^{-t} \end{bmatrix} = \begin{bmatrix} -2e^{-t} \\ -e^{-t} \end{bmatrix}$

LHS = RHS ✓

Verify sol'ns

Verify η_2

Example 2. Given the results from the previous example, solve the IVP with $x(0) = -8$ and $y(0) = 1$.

Verify fundamentalness

$$c_1\eta_1 + c_2\eta_2 = \vec{0}$$

$$\begin{bmatrix} c_1 2e^{-t} + c_2(-1)e^{-6t} \\ c_1 e^{-t} + c_2 2e^{-6t} \end{bmatrix} = \vec{0}$$

$$\hookrightarrow 2c_1e^{-t} = c_2e^{-6t}$$

$$c_2 = 2c_1e^{5t} \qquad c_1e^{-t} + 2(2c_1e^{5t})e^{-6t} = 0$$

$$\rightsquigarrow 5c_1e^{-t} = 0 \to c_1 = 0 \Rightarrow c_2 = 0 \ ✓$$

$$\mathbb{X} = [\eta_1 \ \eta_2] = \begin{bmatrix} 2e^{-t} & -e^{-6t} \\ e^{-t} & 2e^{-6t} \end{bmatrix}$$

$$\vec{\eta} = \mathbb{X}\vec{c}$$

$$x(0) = -8 \qquad y(0) = 1 \Longrightarrow \eta(0) = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$$

$$\rightsquigarrow \eta(t) = \mathbb{X}(t)\vec{c} \Rightarrow \eta(0) = \begin{bmatrix} -8 \\ 1 \end{bmatrix} = \mathbb{X}(0)\vec{c} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\uparrow$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} -8 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -15 \\ 10 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

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Section 3.4

3.4 Eigenvalue Method

Recall Example 1 from how we found the solutions to $x' = -2x + 2y$ and $y' = 2x - 5y$ to be

$$\mathbb{X} = \begin{bmatrix} 2e^{-t} & -e^{-6t} \\ e^{-t} & 2e^{-6t} \end{bmatrix}. \longleftarrow \text{Fundamental Matrix}$$

Linearly independent solutions

Note that we can also write this matrix as $\mathbf{x}_1 = e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = e^{-6t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

A is the coeff. matrix

We have solutions that are of the form

$$\mathbf{x} = e^{rt}\mathbf{u}.$$

How often does this happen? Or rather when is $\mathbf{x} = e^{rt}\mathbf{u}$ a solution for $\mathbf{x}' = A\mathbf{x}$?

$$x' = re^{rt}\vec{u}$$

$$x' = Ax = Ae^{rt}\vec{u}$$

$$\Rightarrow r\vec{u} = A\vec{u}$$

Eigen stuff!

$\vec{u} \neq \vec{0}$ eigenvector

r eigenvalue

$X = e^{rt}\vec{u}$ is a sol'n for the diff eq iff r is an eigenvalue with eigenvector $\vec{u}$

$$\rightsquigarrow (A - rI)\vec{u} = \vec{0}$$

Sol'ns = $\ker(A - rI)$ ↗ $\det(A - rI) = 0$

> **Eigenvalue Method**
> In summary, $\mathbf{x} = e^{rt}\mathbf{u}$ is a solution to $\mathbf{x}' = A\mathbf{x}$ iff $\exists r$ and $\mathbf{u} \neq \mathbf{0}$ such that
>
> $\vec{X} = e^{rt}\vec{u}$ ← constant vector
>
> $$A\mathbf{u} = r\mathbf{u} \text{ or } (A - rI)\mathbf{u} = \mathbf{0}.$$

Note that these only exist for r such that $\det(A - rI) = 0$.

In this case, r is an eigenvalue of A and $\mathbf{u}$ is the eigenvector corresponding to r.

Example 1. $A = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix}$. Find the eigenvalues and eigenvectors!

$$\det(A - rI) = \begin{vmatrix} -2-r & 2 \\ 2 & -5-r \end{vmatrix} = (-2-r)(-5-r) - 4 = 6 + 7r + r^2 \; ; \; r = -6, -1$$

$$r = -1 : \ker(A + I) = \ker\left(\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\right) = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} x_1 & x_2 & \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 = 2x_2 \rightsquigarrow \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$r = -6 : \ker(A + 6I) = \ker\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right) = \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow 2x_1 = -x_2 \rightsquigarrow \vec{u}_6 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\longrightarrow X' = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} X \text{ has solns } X_1 = e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } X_2 = e^{-6t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Example 2. $\mathbf{x}' = A\mathbf{x}$. Given $A = \begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}$ and $\lambda = \pm 3$.

$$\ker(A - \lambda_1 I) = \ker(A - 3I) = \left[\begin{array}{cc|c} -4 & 1 & 0 \\ 8 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 4 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\ker(A - \lambda_2 I) = \ker(A + 3I) = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 8 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow u_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$x = \overset{\mathbb{X}}{\begin{bmatrix} e^{3t} & e^{-3t} \\ 4e^{3t} & -2e^{-3t} \end{bmatrix}} \overset{\vec{c}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

> **Theorem.** If $r_1, \ldots, r_n$ are distinct eigenvalues for $A_{n\times n}$ and $\mathbf{u}_i$ is the eigenvector corresponding to r_i, then $\mathbf{u}_i, \ldots, \mathbf{u}_n$ are linearly independent! Different eigenvalues yield lin. indep. eigenvectors.

$n=2$.

Proof. Sps u_1 has eigenvalue r_1, u_2 has r_2. $^{r_1 \neq r_2}$ Sps BWOC $u_1 = cu_2$. $Au_1 = cAu_2$

$\Rightarrow r_1 u_1 = c r_2 u_2$. $\Rightarrow r_1 u_1 = r_2 u_1 \Rightarrow (r_1 - r_2) u_1 = \vec{0} \Rightarrow r_1 = r_2$ ↯

> **Corollary.** If $r_1, \ldots, r_n$ are distinct eigenvalues for $A_{n\times n}$ and $\mathbf{u}_i$ is the eigenvector corresponding to r_i, then $e^{r_1 t}\mathbf{u}_i, \ldots, e^{r_n t}\mathbf{u}_n$ are linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$!

Example 3. $\mathbf{x}' = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \mathbf{x}$ has a general solution through superposition.

$$x = c_1 \vec{X}_1 + c_2 \vec{X}_2 = [\vec{X}_1 \ \vec{X}_2] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbb{X} \vec{c}$$

That's great, but how do we handle complex roots as a solution to our characteristic polynomial?

Example 4. $\mathbf{x}' = A\mathbf{x}$. Given $A = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$.

Synced ±

$$C_A(\lambda): (-1-\lambda)(-3-\lambda) - 2 \Rightarrow \lambda = -2 \pm i \Rightarrow \vec{u} = \begin{bmatrix} -1 \pm i \\ 1 \end{bmatrix}$$

do the splits!

$$\vec{X} = e^{rt}\vec{u} = e^{(-2+i)t} \begin{bmatrix} -1+i \\ 1 \end{bmatrix} = e^{-2t}(\cos t + i \sin t)\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \underbrace{e^{-2t}\left(\cos t \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}_{x_1} + i \underbrace{e^{-2t}\left(\cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)}_{x_2}$$

Can do the 2nd root + a lin. comb of them to convert $\mathbb{C}$ to $\mathbb{R}$

Example 5. $\mathbf{x}' = A\mathbf{x}$. Given $A = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}$.

Complex Eigenvalues. If $\mathbf{x}(t) = e^{rt}\mathbf{u} = e^{\underline{(\alpha+i\beta)}t}\underline{(\mathbf{a}+i\mathbf{b})}$ is a solution for $\mathbf{x}' = A\mathbf{x}$ with $A \in \mathcal{M}_{2\times 2}$, then

don't confuse a w/ α, β w/ b.

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b} \text{ and} \\ \mathbf{x}_2(t) &= e^{\alpha t}\sin\beta t\mathbf{a} + e^{\alpha t}\cos\beta t\mathbf{b} \end{aligned}$$

are linearly independent solutions.

That's even greater, but how do we handle repeated roots as a solution to our characteristic polynomial?

Example 6. $\mathbf{x}' = A\mathbf{x}$. Given $A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$.

alg deg 2
geom deg 1

$$C_A(\lambda) = (1-\lambda)(-3-\lambda) - 4 \to r = -1 \text{ rep}, \quad u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to X_1 = e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Guess $x_2 = te^{-t}\vec{u}_1 + e^{-t}\vec{u}_2$

$$x_2' = (1-t)e^{-t}u_1 - e^{-t}u_2 = \underline{e^{-t}(u_1-u_2)} + \underline{te^{-t}(-u_1)} \leftarrow \text{Equal}$$

$$A\vec{x}_2 = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}\vec{x}_2 = \underline{te^{-t}\begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}}u_1 + \underline{e^{-t}\begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}u_2}$$

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}u_1 = -u_1}_{\text{Coincidence!}}$$

u_1 is eigenvector from above

$$\begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}u_2 = u_1 - u_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - u_2$$

$$\begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}u_2 + u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(A + I)u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 4 & -2 & 2 \end{array}\right] \rightsquigarrow u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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Section 3.5

3.5 Two dimensional systems and their vector fields

As we saw before, we can make slope fields if we have autonomous ODEs. Suppose our first order system is autonomous:

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}$$

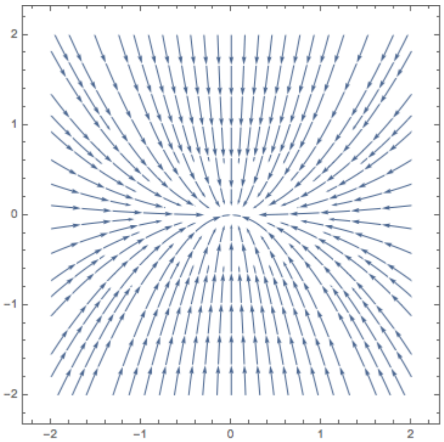
for some functions f and g. Again, note that both x and y are functions of the same independent variable, t. If we look at *just the* (x, y) *plane*, we have

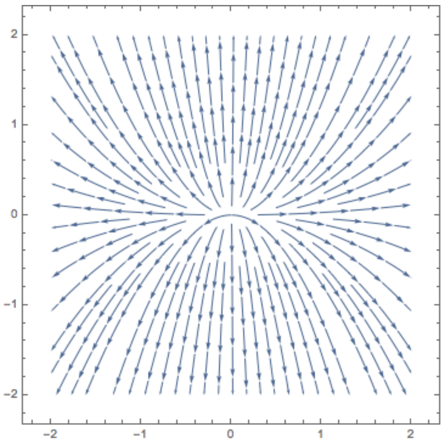
This is important:

Phase planes have two major uses:

1.
2.

Example 1. $\begin{aligned} x' &= -x \\ y' &= -2y \end{aligned}$





Example 2. $\begin{aligned} x' &= x \\ y' &= 2y \end{aligned}$

Example 3.
$$\begin{aligned} x' &= -y(y-2) \\ y' &= (x-2)(y-2) \end{aligned}$$

Equilibia. A point (x_0, y_0) where $x' = y' = 0$ is called a *critical point* or *equilibrium point.* The solution $x(t) = x_0$, $y(t) = y_0$ is called an *equilibrium solution.* The set of all critical points is the *critical set.*

Example 4. Find all critical points in the previous examples.

Example 5.
$$\begin{aligned} x' &= x^2 - 2xy \\ y' &= 3xy - y^2 \end{aligned}$$

Let's assume our autonomous system is also linear and homogeneous, so we have

2-dim, 1st order, linear, homog, const. coeff

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}, \text{ or } \qquad \vec{X}' = A\vec{X} \quad \text{for} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{w/} \det A \neq 0$$

Solutions to $\mathbf{x}' = A\mathbf{x}$ are

so they appear as curves in the phase plane. Equilibria solutions are constant solutions (where all derivatives are 0), so a solution is an equilibrium if and only if

Since ker A always contains $\mathbf{0}$,

When are there other, nontrivial equilibrium solutions?

→ I want to… :(

The det $A = 0$ situation is more complicated (take MA337!), so we'll assume det $A \neq 0$. That is, we're looking at systems of the form $\mathbf{x}' = A\mathbf{x}$, where det $A \neq 0$. Thus,

Commence cases! (Repeated root is not covered in this course)

Case 1: Two distinct real eigenvalues

$\lambda_1 > \lambda_2$, with vectors $\vec{V}_1, \vec{V}_2$

The eigenvectors produce solutions called *eigensolutions*:

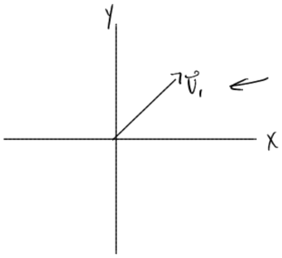
$$\vec{X}_1 = e^{\lambda_1 t}\vec{V}_1 \qquad \vec{X}_2 = e^{\lambda_2 t}\vec{V}_2 \qquad \vec{X} = \overline{X}\,\vec{c}$$

$$\vec{X}_1 = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} v_{11} \\ e^{\lambda_1 t} v_{12} \end{bmatrix} \qquad \Longrightarrow \overline{X} = \begin{bmatrix} e^{\lambda_1 t} v_{11} & e^{\lambda_2 t} v_{21} \\ e^{\lambda_1 t} v_{12} & e^{\lambda_2 t} v_{22} \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \vec{X}_1 \qquad\qquad \vec{X}_2$$

We have ~~$\frac{x(t)}{y(t)} = \frac{v_1}{v_2}$~~, so $\frac{y(t)}{x(t)} = \frac{v_{12}}{v_{11}} = m$

Thus,



← straight line soln given by eigenvector $\vec{V}_1$

Note, since $\det A \neq 0 \Leftrightarrow$ not a degenerate system $\Leftrightarrow \vec{0}$ is only equilibrium

What about other solutions? Note that $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\overset{v_1}{\cancel{\mathbf{u}}} + c_2 e^{\lambda_2 t}\mathbf{v}_2$, so

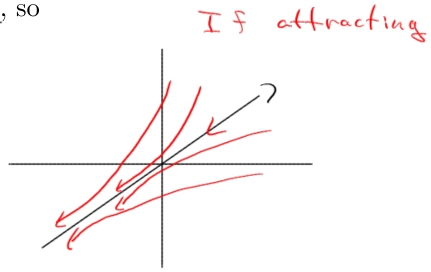
$$\frac{y(t)}{x(t)} = \frac{c_1 e^{\lambda_1 t} v_{12} + c_2 e^{\lambda_2 t} v_{22}}{c_1 e^{\lambda_1 t} v_{11} + c_2 e^{\lambda_2 t} v_{21}} = \frac{v_{12} + \frac{c_2}{c_1} e^{(\lambda_2 - \lambda_1)} v_{22}}{v_{11} + \frac{c_2}{c_1} e^{(\lambda_2-\lambda_1)} v_{21}}$$

$\lambda_1 > \lambda_2$ so $t \to \infty$ forces $e^{(\lambda_2 - \lambda_1)t} \to 0$

Since $\lambda_2 - \lambda_1$, we can

$$\lim_{t\to\infty} \frac{y(t)}{x(t)} = \frac{v_{12}}{v_{11}} \qquad !$$

If $\lambda_2 < \lambda_1 < 0$, then $x(t)$ and $y(t)$ both decay exponentially, so



Thus,

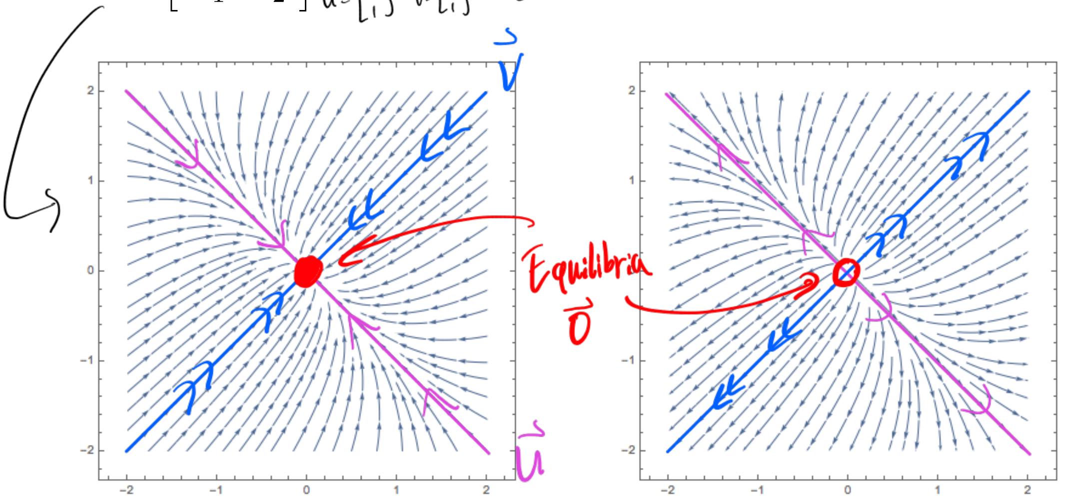
> **Stable and Unstable Nodes.** If the eigenvalues of $A \in \mathcal{M}_{2\times 2}$ are real, distinct, and negative (positive), then the phase plane of $\mathbf{x}' = A\mathbf{x}$ is called a *stable (unstable) node* and the origin is an *attractor (repeller)*.

Fun fact: Stable: $\lambda_2 < \lambda_1 < 0$

$x_1 \to 0$ as $t \to \infty$

$x_2 \to 0$ as $t \to \infty$

Example 6. $A = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$ $\lambda_1 = -1$, $\lambda_2 = -3$ $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\leftarrow$ $|\lambda_1| < |\lambda_2|$ so v has a "stronger" pull/push



Unstable: $0 < \lambda_2 < \lambda_1$

Example 7. $A = \begin{bmatrix} +2 & +1 \\ +1 & +2 \end{bmatrix}$

$\lambda_1 = 1$ $\quad \vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$ $\quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

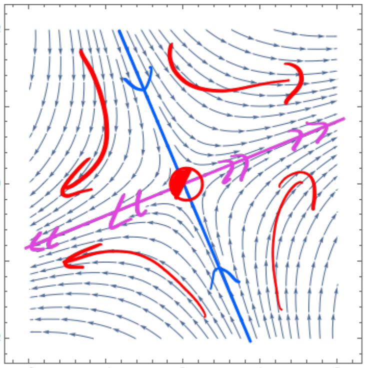
$x_1 \to \infty$ as $t \to \infty$

$x_2 \to \infty$ as $t \to \infty$

Case 1b: Two distinct real eigenvalues and $\lambda_1 < 0 < \lambda_2$

Stable and Unstable manifolds. If the eigenvalues of $A \in \mathcal{M}_{2\times 2}$ are $\lambda_1 < 0 < \lambda_2$, then the eigensolution associated to $\lambda_1 < 0$ is called the *stable manifold*. The eigensolution associated to $\lambda_2 > 0$ is called the *unstable manifold*. The associated phase plane is called a *saddle node*.

Example 8. $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$



$\vec{v}$ unstable node/manifold

$\lambda = \pm\sqrt{2}$

$\vec{u}$ stable node/manifold

Example 9. $\begin{aligned} x' &= by \\ y' &= cx \end{aligned}$ with $b, c > 0$

Parameters!
Scalars!

$\vec{X}' = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ $\det(A - \lambda I) = \lambda^2 - bc$

$\lambda = \pm\sqrt{bc}$, saddle-type behavior

Case 2: Complex eigenvalues

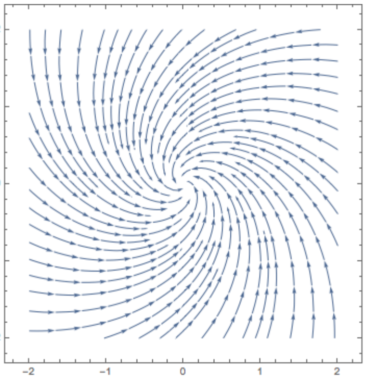
If $A \in \mathcal{M}_{2\times 2}$, has eigenvalue $\lambda = \alpha \pm i\beta$ with $\beta \neq 0$ and associated eigenvector $\mathbf{a} + i\mathbf{b}$, then

Seperate real and imaginary components

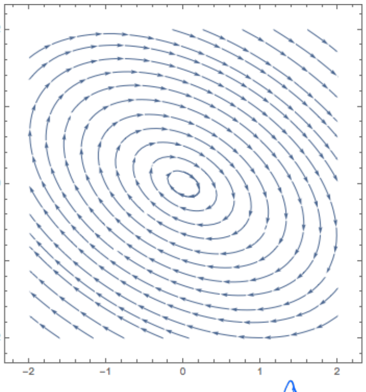
$\vec{a}, \vec{b} \in \mathbb{R}^2$

$$\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 = c_1 e^{\alpha t}(\cos\beta t\,\vec{a} - \sin\beta t\,\vec{b}) + c_2 e^{\alpha t}(\sin\beta t\,\vec{a} - \cos\beta t\,\vec{b})$$
$$= e^{\alpha t}\begin{bmatrix} (c_1a_1 + c_2b_1)\cos\beta t + (c_2a_1 - c_1b_1)\sin\beta t \\ (c_1a_2 + c_2b_2)\cos\beta t + (c_2a_2 - c_1b_2)\sin\beta t \end{bmatrix}$$

If $\alpha = 0$, then ↗ is the parametric eqn of an ellipse! There are three subcases:

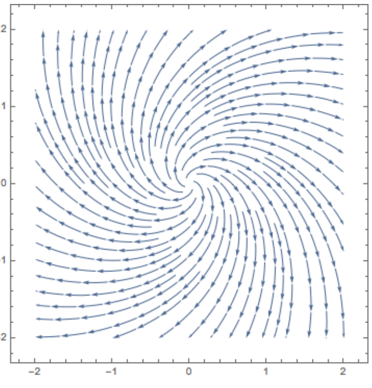


$\alpha < 0$



$\alpha = 0$ Center

Check pts to test orientation



$\alpha > 0$

Stable spiral

Unstable spiral

Centers and Spirals. If the eigenvalues of $A \in \mathcal{M}_{2\times 2}$ are $\lambda = \alpha \pm i\beta$ with $\beta \neq 0$, then the associated phase plane is called a *stable spiral* when $\alpha < 0$, a *center* when $\alpha = 0$, and an *unstable spiral* when $\alpha > 0$.

If $\alpha \neq 0$, then $\mathbf{x}(t)$ is

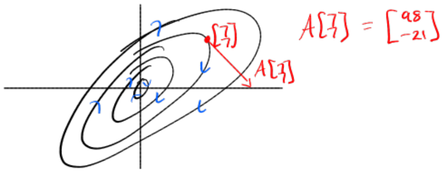
Note: orientation of a spiral (clockwise or counterclockwise) or direction on ellipses is not clear from eigen-stuff. *You must test a point!*

Example 10. $A = \begin{bmatrix} 0 & -4.34 \\ 0.208 & -0.078 \end{bmatrix}$ has $\lambda = -0.039 \pm 0.949i$ as an eigenvalue. Thus,

$\alpha < 0 \to$ Stable!

Note that $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, so

Example 11. $A = \begin{bmatrix} 1 & 13 \\ -2 & -1 \end{bmatrix}$ $\lambda = \pm 5i$ barf!



Let's put this all into a convenient chart!

Eigen values	Phase plane
2 real >0	Unstable node
2 real <0	Stable node (horses)
2 real $\lambda_2 < 0 < \lambda_1$	Saddle
Pure imaginary ($a=0$)	Center
Complex, Re[z]>0	Unstable spiral
Complex, Re[z]<0	Stable spiral

Dr Kaschner, 24 Oct 24

"I wake up, hit my head against the wall three times, and think of $\mathbb{R}^4$"

Example 12.
$$\begin{aligned} x'' &= -3x + y \\ y'' &= 2x - 2y \end{aligned}$$

$W = X'$
$W' = X''$
$Z = Y'$
$Z' = Y''$

$$\vec{X} = \begin{bmatrix} X \\ W \\ Y \\ Z \end{bmatrix} \rightsquigarrow \vec{X}' = \begin{bmatrix} W \\ -3X + Y \\ Z \\ 2X - 2Y \end{bmatrix} = \overset{\mathcal{A}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}} \begin{bmatrix} X \\ W \\ Y \\ Z \end{bmatrix}$$

$\lambda = \pm i, \pm 2i$

Mathematica due Tuesday

Portfolio due 11/5

Project

Math 334 – Differential Equations

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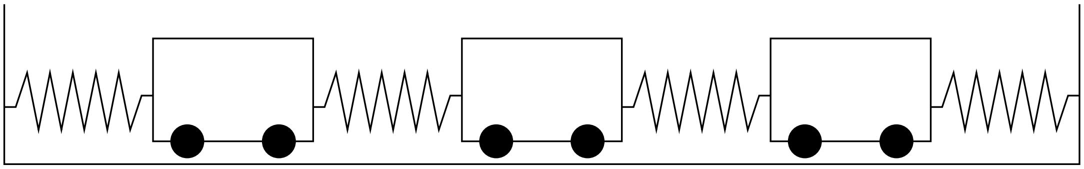
Section 3.6

3.6 ~~Two Dimensional Systems Applications~~ Bonus Springs

Example 1. Here is a example model. How can we turn this into a system of equations and solve it?



First, let's pretend one of these carts isn't really there.

Hooke's Law

$F_{spring} = -kx$

F force

x displacement

k spring constant >0

Newton's 2nd Law

$F = ma$

$= mx''$

m = mass >0

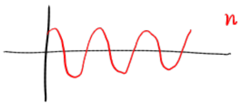
a = acceleration = x''

$\rightarrow mx'' + kx = 0$

Guess $x = e^{rt} \rightsquigarrow x = c_1 \cos\sqrt{\frac{k}{m}}t + c_2 \sin\sqrt{\frac{k}{m}}t$

$= \sqrt{c_1^2+c_2^2} \cos\left(\sqrt{\frac{k}{m}}t + \arctan\frac{c_1}{c_2}\right)$

$\rightsquigarrow$



no dampening!

Encorperate friction

b>0 friction constant

$mx'' + bx' + kx = 0$

$x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$

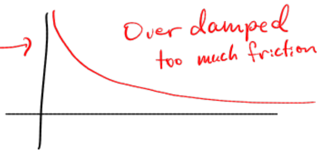
$x = e^{rt} \rightsquigarrow r^2 + \frac{b}{m}r + \frac{k}{m} = 0$

$r = \frac{-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - \frac{4k}{m}}}{2} = \frac{-b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$

Real soln iff $\Delta > 0 \iff b^2 > 4km$

Imaginary: $x = Ce^{\frac{-b}{m}t} \cos\left(\sqrt{\frac{k}{m}}t + \gamma\right)$ for $c = \sqrt{c_1^2+c_2^2}$, $\gamma = \arctan\left(\frac{c_1}{c_2}\right)$

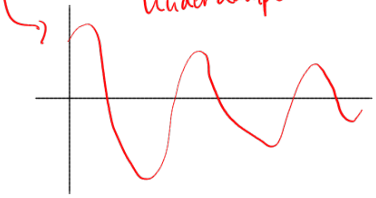
Real: $x = c_1e^{-r_1t} + c_2e^{-r_2t}$



Over damped

too much friction

Underdamped



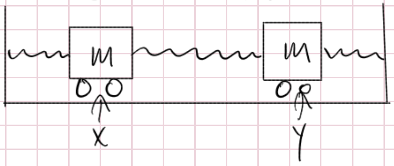
subject

date

keywords

topic Springs, Continued

Carts on wheels



Shared mass m

x, y: displacement from rest

$$\left.\begin{array}{l} mx'' = -kx + k(y-x) \\ \qquad\quad\ \uparrow \qquad\qquad \uparrow \\ \quad\ \text{pull from left} \quad \text{pull from right} \\ \qquad\quad\ \downarrow \qquad\qquad \swarrow \\ my'' = k(x-y) - ky \end{array}\right\} \text{2 equs of second order}$$

$$x' = w \qquad w' = x'' = \frac{-2k}{m}x + \frac{k}{m}y$$

$$y' = z \qquad z' = y'' = \frac{k}{m}x - \frac{2k}{m}y$$

$$\vec{x} = \begin{bmatrix} x \\ y \\ w \\ z \end{bmatrix} \qquad \vec{x}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-2k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & \frac{-2k}{m} & 0 & 0 \end{bmatrix}$$

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Section 3.9.2

3.9.2 Var of Parm for Nonhomogeneous Systems

There are a lot of really cool things in Section 3.9. Alas, this is all we have time to cover.

Consider the nohomogeneous, nonautonomous linear system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f},$$

where $A \in \mathcal{M}_{n\times n}(C^1(\mathbb{R}))$.[1] As one might expect, the general solution is of the form

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

or $\vec{x}_h$

$\vec{x}_h$ soln to homogeneous part

$\vec{x}_p$ soln (only one!) to non-homog part

$\vec{x}_h(t) = \mathbb{X}(t)\vec{c}$

As you surely expect, we'll use var of parm to get $\mathbf{x}_p$. We'll guess

Recall: $x'' = f$, guess $x_p = c(t)x_1(t)$

$\vec{x}_p(t) = \mathbb{X}(t)c(t)$

↳ plug in to $\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t)$

LHS RHS

LHS: $\vec{x}_p' = \mathbb{X}'(t)c(t) + \mathbb{X}(t)c'(t)$

RHS: $\vec{x}_p' = A\mathbb{X}(t)c(t) + f(t)$

$$\mathbb{X} = [\vec{x}_1 \ \vec{x}_2 \ \cdots \ \vec{x}_n] \longrightarrow \mathbb{X}' = [A\vec{x}_1 \ A\vec{x}_2 \ \cdots \ A\vec{x}_3] = A\mathbb{X}$$

All of these are fxns of t

$$\hookrightarrow \cancel{A\mathbb{X}c} + \mathbb{X}c' = \cancel{A\mathbb{X}c} + f$$

$$f(t) = \mathbb{X}(t)c'(t)$$

$$c'(t) = \mathbb{X}^{-1}(t)f(t)$$

Thus,

$$\boxed{c(t) = \int \mathbb{X}^{-1}(t)f(t)\,dt}$$

$$\vec{x} = \mathbb{X}\vec{c} + \mathbb{X}\int \mathbb{X}^{-1}f\,dt$$

---

[1] These are just $n \times n$ matrices whose entries are continuously differentiable functions of the independent variable (probably t).

[2] WHY?

Quiz 5 Thursday (date change)

Portfolio 2A Tuesday

Var of Parm for Systems. If $A(t)$ and $\mathbf{f}(t)$ are continuous in some interval I, then

$$\mathbf{x}(t) = \mathbb{X}(t)\mathbf{c} + X(t)\int \mathbb{X}^{-1}(t)\mathbf{f}(t)\,dt,$$

is the general solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Example 1. Solve $x'' + x = \cos 2t$ system-style.

Ch 2: Homog: $x''+x=0 \rightsquigarrow r=\pm i \rightsquigarrow X_1 = \cos t + i\sin t \quad X_2 = \cos t - i\sin t$

Now: let $y = x' \to y' = x'' = -x + \cos 2t \Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x + \cos 2t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \cos 2t \end{bmatrix}$

Homog: $\vec{x}' = A\vec{x}$ w/ goal: $\vec{x} = \vec{x}_h + \vec{x}_p$.

Guess: $\vec{x} = e^{rt}\vec{u} \Longrightarrow A\vec{u} = r\vec{u} \Rightarrow \begin{matrix} r & \text{e-val} \\ \vec{u} & \text{e-vec} \end{matrix} \rightsquigarrow r = \pm i,\ u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Consistent $\overset{\pm}{\longleftrightarrow}$

$r = \alpha + i\beta = 0 + i$

$\vec{u} = \vec{a} + i\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{x}_c = c_1\left(\cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + c_2\left(\sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\mathbb{X}$ hey, not bad!

Var of parm:

$$x_p = \mathbb{X}\int \mathbb{X}^{-1} f(t)\,dt = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\int \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \cos 2t \end{bmatrix} dt$$

$\det \mathbb{X} = \cos^2 t + \sin^2 t = 1$!!

$$\vec{x}_p = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\int \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 0 \\ \cos 2t \end{bmatrix} dt$$

$$= \text{—//—} \int \begin{bmatrix} -\sin t\cos 2t \\ \cos t\cos 2t \end{bmatrix} dt$$

$$= \text{—//—} \begin{bmatrix} \int -\cos 2t \sin t\,dt \\ \int \cos t\cos 2t\,dt \end{bmatrix}$$

Fun fact $\cos 2t = 2\cos^2 t - 1$

$$= \text{—//—} \begin{bmatrix} \frac{2}{3}\cos^3 t - \cos t \\ \sin t - \frac{2}{3}\sin^3 t \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3}\cos 2t \\ \frac{1}{3}\sin 2t \end{bmatrix}$$

$$x = x_c + x_p = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3}\cos 2t \\ \frac{1}{3}\sin 2t \end{bmatrix}$$

Here's s fundamental fact[3] you may have forgotten: If f is continuous on $[a, b]$ and

$$F(t) = \int_a^t f(s)\,ds,$$

also recall $c(t) = \int \mathbb{X}^{-1} f\,dt$

then $F'(t) = f(t)$ on $[a, b]$. In particular, when we're defining $\mathbf{c}(t)$ on $[a, b]$, we should[4] really be writing

$$c = \int_a^t \mathbb{X}^{-1} f\,ds \quad \text{for } c(t) \text{ on } [a, b]$$

---

[3] Theorem.

[4] If Newton saw what we did before, he'd probably make the ghost of Leibniz haunt us.

Not Found in Lebl

Let's look at the nonhomogeneous IVP

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0.$$

We know the general solution is

$$\vec{x} = \mathbb{X}\vec{c} + \mathbb{X}\int_{t_0}^{t} \mathbb{X}^{-1} F \, ds$$

Note that we've chosen to start our integral at x_0. The Fundamental Theorem of Calculus let's us choose, and this is a good choice. Look what happens when we apply the initial condition: and solve for $\vec{c}$.

$$X_0 = X(t_0) = \mathbb{X}(t_0)\vec{c} \; \underbrace{\mathbb{X}(t_0)\int_{t_0 \leftarrow \text{always } t_0}^{t_0 \leftarrow \text{our Variable bound}} \mathbb{X}^{-1} F(t_0)\, ds}_{0}$$

Thus, we have

$$\vec{X}_0 = \mathbb{X}(t_0)\vec{c}$$

so

$$\boxed{\vec{C} = \mathbb{X}^{-1}(t_0)\vec{X}_0}$$

Example 2. Solve
$$\begin{aligned} x' &= -2x + 2y \overset{+}{\cancel{}} e^{-2t} \\ y' &= 2x - 5y \end{aligned} \quad \text{w/ } X_0(0) = \begin{bmatrix} 1/10 \\ 21/20 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \quad \lambda = -1, -6 \quad u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\mathbb{X} = \begin{bmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{bmatrix} \qquad \mathbb{X}^{-1} = \frac{-1}{5e^{-7t}}\begin{bmatrix} 2e^{-6t} & e^{-6t} \\ e^{-t} & -2e^{-t} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2e^{t} & e^{t} \\ e^{6t} & -2e^{6t} \end{bmatrix}$$

$$\vec{x} = \mathbb{X}\vec{c} + \mathbb{X}\int \mathbb{X}^{-1}\vec{f}\, dt$$

$$\vec{C} = \mathbb{X}^{-1}(t_0)x_0 = \mathbb{X}^{-1}(0)\begin{bmatrix} \frac{1}{10} \\ 21/20 \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & -2/5 \end{bmatrix}\begin{bmatrix} 1/10 \\ 21/20 \end{bmatrix} = \begin{bmatrix} \frac{4}{100} + \frac{21}{100} \\ \frac{2}{100} - \frac{42}{100} \end{bmatrix} = \begin{bmatrix} 1/4 \\ -2/5 \end{bmatrix}$$

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Section 6.1

Integrable Functions	I	Functions
$f : \mathbb{R} \to \mathbb{R}$	$\Rightarrow$	$g : \mathbb{R} \to \mathbb{R}$

An *operator* is just a linear transformation on a vector space of functions (like $\mathbb{P}_n$ or $C^1([a,b])$). Oh. In case you forgot, $C^1([a,b])$ is the set of functions defined on the interval $[a,b]$ that have continuous first derivative. It's totally a vector space. You should check. Here's an integral operator:

$$I \colon C^1([a,b]) \to C^1([a,b]) \text{ by}$$
$$I(f) = \int f(x)\,dx$$

You should verify that I is a linear transformation. When your done, you should be sad that you can't make a nice matrix representation for I because $C^1([a,b])$ is infinite dimensional. Sorry. That is very sad.

Hey! Define $I \colon \mathbb{P}_n \to \mathbb{P}_{n+1}$ by $I(f) = \int p(x)\,dx$. There. Now you can make a matrix representation for I.

6.1 Laplace Transform

$$\mathcal{L}(f(t)) := \int_0^\infty f\, e^{-st}\, dt$$

This is a function of s. People usually use capital letters for Laplace transformed functions:

$$F(s) = \mathcal{L}\{f(t)\}.$$

Why is this relevant to Differential Equations?

-
-
-

Let's do an example!

-

Example 1. $\mathcal{L}\{1\}$

$$\mathcal{L}\{1\} \overset{\triangle}{=} \int_0^\infty e^{-st}\,dt = \frac{-1}{s}\Big|_0^\infty = \frac{1}{s}$$

Another one.

Example 2. $\mathcal{L}\{t\}$

$$\mathcal{L}\{t\} \overset{\circ}{=} \int_0^\infty te^{-st}\,dt \qquad \begin{array}{ll} +t & e^{-st} \\ -1 & \frac{-1}{s}e^{-st} \\ +0 & \frac{1}{s^2}e^{-st} \end{array} \qquad = \left[\frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^\infty = \frac{1}{s^2}$$

Another one.

Example 3. $\mathcal{L}\left\{e^{-3t}\right\}$

$$\mathcal{L}\{e^{-3t}\} = \int_0^\infty e^{-3t}e^{-st}\,dt = \int_0^\infty e^{-(s+3)t} = \frac{1}{s+3}$$

Common Laplace Transforms!

- $$\mathcal{L}\{1\} = \frac{1}{s}$$
- $$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$
- $$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$
- $$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$
- $$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

Laplace transform Fun Facts! The Laplace transform is a linear operator!

$$\mathcal{L}\left\{\alpha f(t)+\beta g(t)\right\}=\alpha\mathcal{L}\left\{f(t)\right\}+\beta\mathcal{L}\left\{g(t)\right\}$$

Example 4. $\mathcal{L}\left\{3t-5\sin(2t)\right\}$

$= 3\,\mathcal{L}\{t\} - 5\,\mathcal{L}\{\sin 2t\}$

Inverse Laplace Transform

$$\mathcal{L}^{-1}\left\{F(s)\right\}$$

What are some cool things about the Inverse Laplace Transform?

-
-

Example 5. $\mathcal{L}^{-1}\left\{\frac{4}{s}+\frac{6}{s^5}-\frac{1}{s+8}\right\}$

Example 6. $\mathcal{L}^{-1}\left\{\left(\frac{2}{s}-\frac{1}{s^3}\right)^2\right\}$

Example 7. $\mathcal{L}^{-1}\left\{\frac{10s}{s^2+16}\right\}$

Example 8. $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\}$

First Translation Theorem Let a be any real number. Let $F(s)$ denote $\mathcal{L}\{f(t)\}$. Then,

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$$

Reason:

Example 9. $\mathcal{L}\left\{e^{7t}t^3\right\}$

Example 10. $\mathcal{L}\left\{e^{-2t}\cos(4t)\right\}$

Example 11. $\mathcal{L}^{-1}\left\{F(s-a)\right\}$

Example 12. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$

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Section 6.2

Derivatives of Transforms:
$n = 1, 2, 3, ..$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Example 1. $\mathcal{L}\{t^2 \sin(kt)\}$

$$= k(s^2+k^2)^{-1}$$

$$= \cancel{(-1)^2} \frac{d^2}{ds^2} \mathcal{L}\{\sin kt\} = \frac{d^2}{ds^2}\left[k/_{s^2+k^2} \right] = \frac{d}{ds}\left[k(-1)(s^2+k^2)^{-2}(2s) \right]$$

$$= \cdots = \frac{6ks^2 - 2k^3}{(s^2+k^2)^3}$$

How does the Laplace Transform of Derivatives work?

$$\mathcal{L}\{f'\} \overset{\triangle}{=} \int_0^\infty f' e^{-st}\,dt \quad \begin{array}{cc} + \; e^{-st} & f' \\ - \; -se^{-st} & f \\ s^2e^{-st} & \int f \end{array} \quad = e^{-st} f \Big|_0^\infty + s\int_0^\infty e^{-st} f\,dt = -f(0) + s\overset{\mathcal{L}\{f\}}{\overset{\shortparallel}{F(s)}}$$

$$\mathcal{L}\{f''\} \overset{\triangle}{=} \int_0^\infty f'' e^{-st}\,dt = \cdots = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

But what about the Laplace Transform of a second derivative?

See above ...

Okay that's pretty neat. I think I see a pattern; can we generalize this?

$$y'' - 4y' + 4y = 1$$

2nd order linear nonhomog DE

Only works w/ IVPs!

Example 2. Find $\mathcal{L}\{y'' - 4y' + \overset{4}{\text{[illegible]}}y\}$ given $y(0) = 1$ and $y'(0) = -1$.

$$= \mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\}$$

$$= \left(s^2 Y - s\,\overset{1}{\cancel{y(0)}} - \overset{-1}{\cancel{y'(0)}}\right) - 4\left(sY - \overset{1}{\cancel{y(0)}}\right) + 4Y$$

$$= Y(s^2 - 4s + 4) - s + 5$$

Sps $y'' - 4y + y = 1$

$$\Rightarrow Y(s^2 - 4s + 4) - s + 5 = \frac{1}{s}$$

$$Y(\underline{\quad}) = \frac{1}{s} + s - 5$$

$$Y = \frac{\frac{1}{s} + s - 5}{(s-2)^2} = \frac{1}{s(s-2)^2} + \frac{s}{(s-2)^2} - \frac{5}{(s-2)^2}$$

Partial Fractions $\Vert$

$$\boxed{\frac{1/4}{s} + \frac{3/4}{s-2} + \frac{-5/2}{(s-2)^2}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s} + \frac{3}{4}\frac{1}{s-2} - \frac{5}{2}\frac{1}{(s-2)^2}\right\}$$

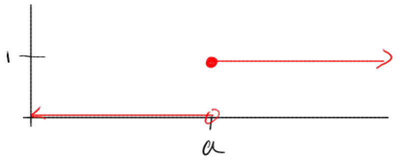
$$\boxed{y = \frac{1}{4} + \frac{3}{4}e^{2t} - \frac{5}{2}te^{2t}}$$

Unit Step Function:

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

How does the Unit Step function interact with another function?

Characteristic on $[a, \infty)$



Second Translation Theorem:

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

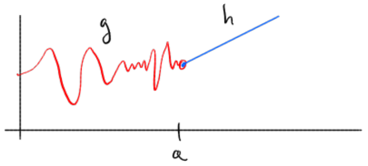
$$= e^{-as}\mathcal{L}\{f(t)\}$$

Example 3. $\mathcal{L}\{\sin(t)\mathcal{U}(t-2\pi)\}$

$$= e^{2\pi s}\mathcal{L}\{\sin(t+2\pi)\} = e^{-2\pi s}\mathcal{L}\{\sin t\}$$

$$= e^{-2\pi s}/_{s^2+1}$$

$$f(x) = \begin{cases} g(x) & x < a \\ h(x) & x \geq a \end{cases}$$



$$f(x) = h(x)\mathcal{U}(t-a) + g(x)(1 - \mathcal{U}(t-a))$$

Portfolio 2B due Fri 11/15
Test 2 Thursday 11/21 Ch 3/6

Example 4. $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/2}}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3e^{-\pi s/2}}{s^2+9}\right\}_{t\to t-\frac{\pi}{2}} = \sin(3t)_{t\mapsto t-\frac{\pi}{2}}\,\mathcal{U}\left(t-\frac{\pi}{2}\right)$

$$= \sin\left(3t-\frac{3\pi}{2}\right)\mathcal{U}\left(t-\frac{\pi}{2}\right)$$

Example 5. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\} = \mathcal{U}(t-2)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\}_{t\to t-2}$

$$= \mathcal{U}(t-2)\,\mathcal{L}\left\{\frac{1}{s-1}-\frac{1}{s}-\frac{1}{s^2}\right\}_{t\to t-2}$$

$$= \mathcal{U}(t-2)\left(e^{t}-1-t\right)_{t\to t-2}$$

$$\boxed{= \mathcal{U}(t-2)\left(e^{t-2}-t+1\right)}$$

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Section 6.3

Example 1. $\mathcal{L}^{-1}\left\{\frac{2}{s^5+s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^3}\left(\frac{1}{s^2+1}\right)\right\} = t^2 * \sin(t) = \int_0^t \tau^2 \sin(t-\tau)\,d\tau$

$$\overset{\text{IBP}\times 2}{=} t^2 - 2 + 2\cos t$$

The Convolution operation $(*)$ is defined for two functions f, g that are piecewise continuous on $[0, \infty)$ as

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Example 2. Given $f(t) = e^t$ and $g(t) = \sin(t)$. Find $f * g$.

$$(f * g)(t) = \int_0^t e^\tau \sin(t-\tau)\,d\tau$$

$$= e^\tau \sin(t-\tau)\Big|_{\tau=0}^{t} + e^\tau \cos(t-\tau)\Big|_{\tau=0}^{t} - \int_0^t e^\tau \sin(t-\tau)\,d\tau$$

$$\Rightarrow (f * g)(t) = \frac{1}{2}\left(e^t - \sin t - \cos t\right)$$

Fun Facts!

- $(cf) * g = f * (cg) = c(f * g)$
- $(f * g) * h = f * (g * h)$
- $f * g = g * f$

Funnest Fact~~!~~s!

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s).$$

which implies

$$\mathcal{L}^{-1}\{F(S)G(S)\} = f*g.$$

Example 3. Find (in terms of t) $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+4)}\right\}$ using Convolution.

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{t} * e^{-4t}$$

$$= \int_0^t e^{\tau} e^{\overset{-4t+4\tau}{-4(t-\tau)}} d\tau = \int_0^t e^{5\tau} e^{-4t} d\tau = \frac{1}{5} e^{5\tau} e^{-4t}\Big|_0^t = \boxed{\frac{1}{5}e^{t} - \frac{1}{5}e^{-4t}}$$

Example 4. Let's figure out $\mathcal{L}\{\int_0^t \cos(\tau)d\tau\}$ $= \mathcal{L}\{\cos t * 1\} = \mathcal{L}\{1\}\mathcal{L}\{\cos t\}$

$$\left.\begin{aligned} f &= \cos t \\ g(t-\tau) &= 1 \end{aligned}\right\} \nearrow$$

$$= \frac{1}{s}\left(\frac{s}{s^2+1}\right) = \frac{1}{s^2+1}$$

This answer looks pretty familiar right?

Note the power of making $g(t) = 1$! In general, we will get

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}.$$

Example 5. Computer Visualization Time! What is the convolution really? Before we answer that. Let's think about what Integrals are. They help us find the area under a curve. Okay so we are looking at the area of something. So what is going on in the integrand? Our f stays the same and then we multiply by a g that is being shifted by t.

As we look at the The Boxes, we are performing $f * g$. Our f is the blue box in the center that is staying in place. The moving red graph is the g. As t moves the area shared between the two graphs changes. The black line tracks the value of the integral at each of these locations. All of these values in aggreagate form the convolution $f * g$

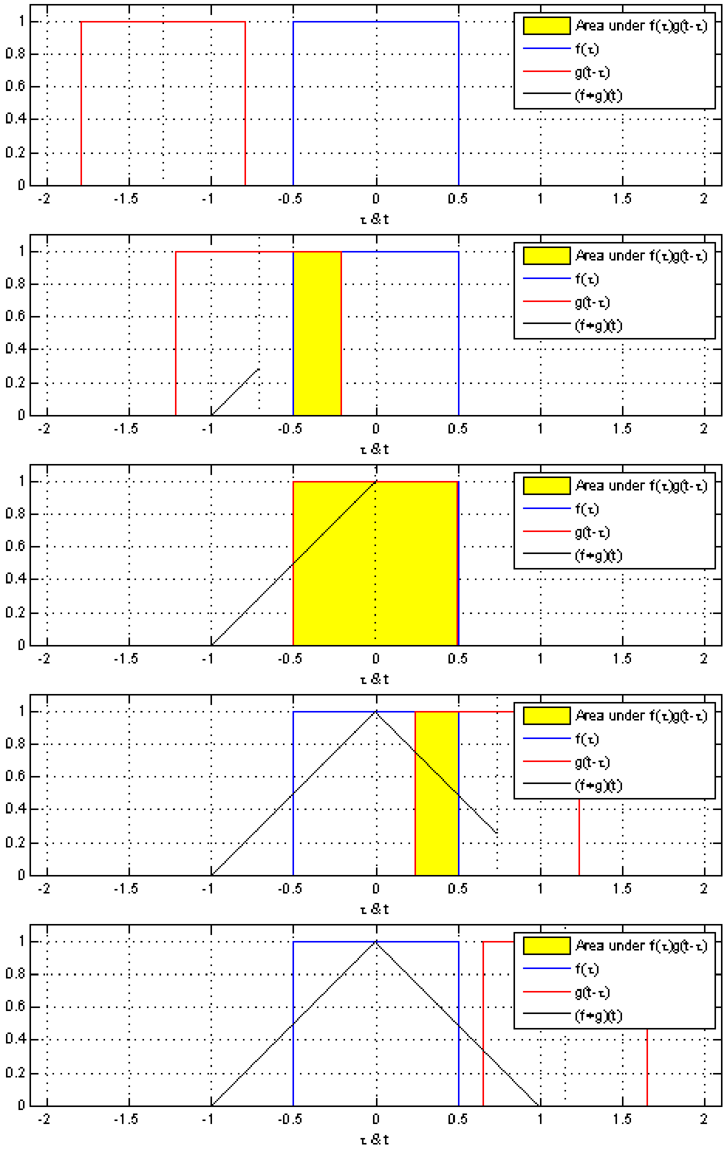


Figure 1: The Boxes

As we look at image 2, we see going down the left column then the right step by step what the convolution looks like graphically. We have two functions $x(t)$ and $h(t)$. However, we need to compose one of these with $-t$ to get $h(-t)$ We see the two graphs overlaid with different t_i's. All of the these t_i's help us form the function generated by $x * h$

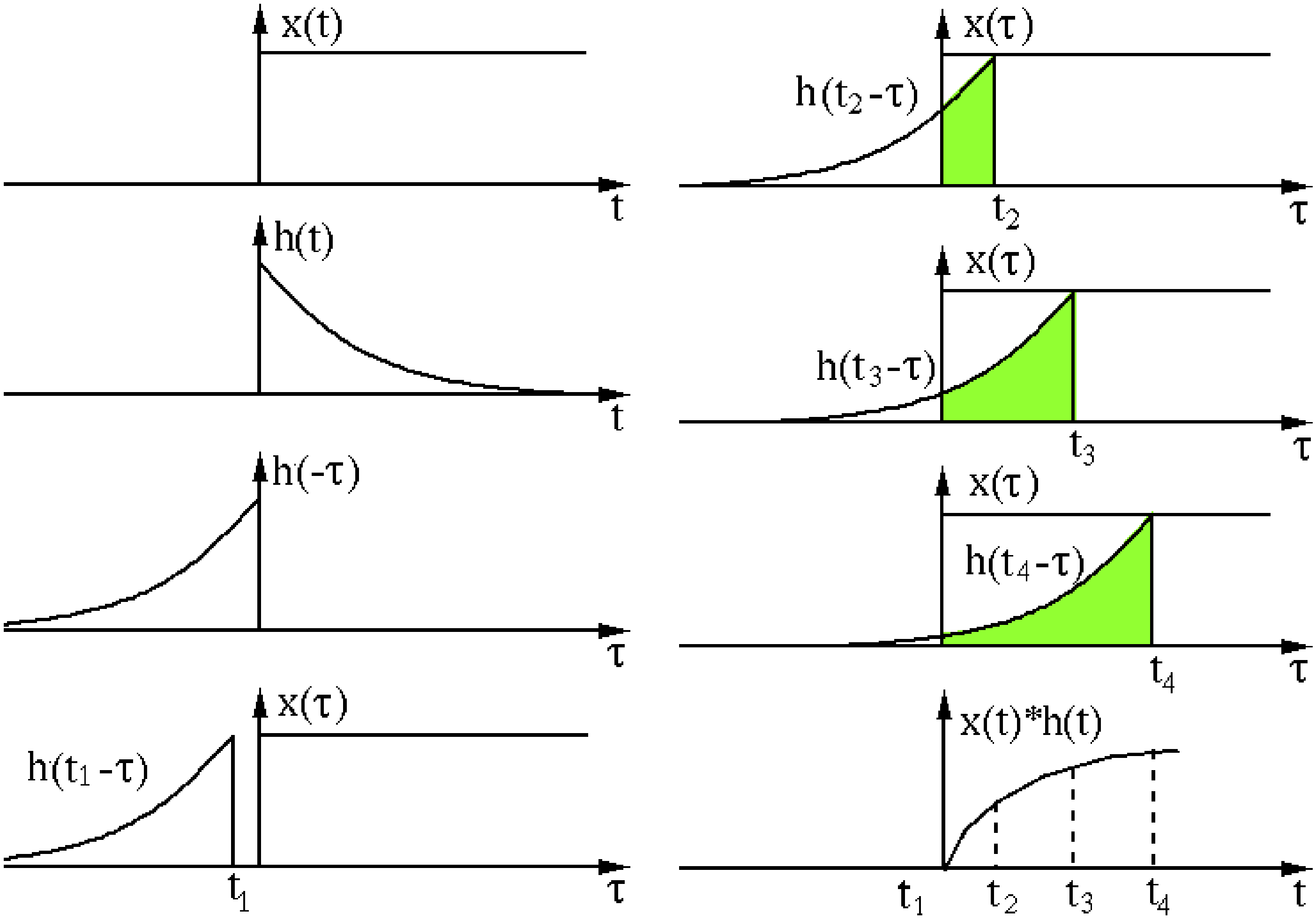


Figure 2: image 2

As we look at image 3, we are seeing $f * g$ and $g * f$ vertically sliced. It highlights the value of the convolution stays the same regardless of the order.

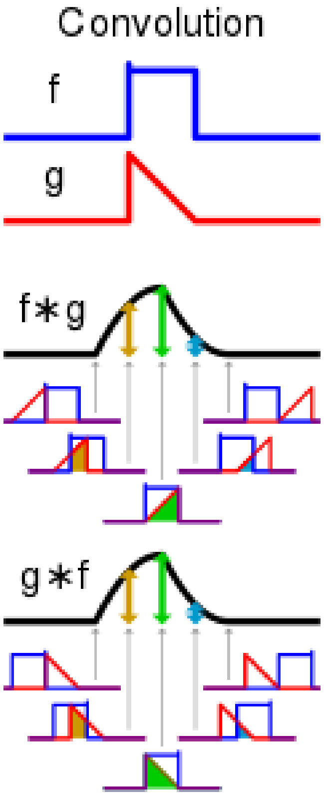


Figure 3: image 3

Example 6. Let's do some Mathematica examples
LaplaceTransform Documentation UnitStep Documentation Convolve Documentation

```
LaplaceTransform[t^4 Sin[t], t, s];
LaplaceTransform[E^(-t), t, s];

Plot[UnitStep[t] , {t,-10,10}];


Convolve[Cos[t]UnitStep[t],t^3UnitStep[t],t,y];
Convolve[Sin[t]UnitStep[t],t^2UnitStep[t],t,y];
```

6.4 Dirac Delta & Impulse Response

$$\int_a^b \delta(t)\,dt = \begin{cases} 1 & 0 \in [a,b) \\ 0 & \text{else} \end{cases}$$

Such that $\int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$

Note $\delta * f = f * \delta = f$, so $\delta = id_*$

$\mathcal{L}\{\delta(t-a)\} = e^{-as}$, $\mathcal{L}\{\delta(t)\} = e^{-0s} = 1$

Ex $x'' + \omega^2 x = \delta(t)$

Only Solvable using Laplace

$$\delta(t-a) = \begin{cases} 0 & t \neq a \\ \infty & t = a \end{cases}$$

$$\boxed{x(0) = x'(0) = 0}$$

Ex $x'' + w^2 x = \delta(t)$

$\mathcal{L}\{x''\} + w^2 \mathcal{L}\{x\} = \mathcal{L}\{\delta(t)\}$

$\mathcal{L}\{x\}s^2 - s\,x(0) - x'(0) + w^2 \mathcal{L}\{x\} = 1$

$\rightsquigarrow \mathcal{L}\{x\}(s^2 + w^2) = 1$

$\mathcal{L}\{x\} = \frac{1}{s^2+w^2} \Rightarrow x = \frac{1}{\omega}\sin \omega t$

$x' + 3x + y' = 1$ $x(0) = 0$

$x' - x + y' - y = e^t$ $y(0) = 0$

$$\mathcal{L}\{ \text{—〃———〃—} \} = \begin{cases} \mathcal{L}\{x' + 3x + y'\} = \mathcal{L}\{1\} \\ \mathcal{L}\{x' - x + y' - y\} = \mathcal{L}\{e^t\} \end{cases}$$

$$\longrightarrow \begin{cases} sF - \cancel{x(0)} + 3F + sG \cancel{-y(0)} = \frac{1}{s} \\ sF - \cancel{x(0)} - F + sG - \cancel{y(0)} - G = \frac{1}{s-1} \end{cases}$$

$$\longrightarrow \begin{cases} (s+3)F + sG = \frac{1}{s} \\ (s-1)F + (s-1)G = \frac{1}{s-1} \end{cases}$$

$\hookrightarrow F = \frac{1}{(s-1)^2} - G$

$(s+3)\left(\frac{1}{(s-1)^2} - G\right) + sG = \frac{1}{s}$

$\frac{s+3}{(s-1)^2} - G(s+3) + s(G) = \frac{1}{s}$

$-3G = \frac{1}{s} - \frac{s+3}{(s-1)^2}$

$G = -\frac{1}{3}\frac{1}{s} + \frac{1}{3}\frac{s+3}{(s-1)^2} \rightsquigarrow 3y = \mathcal{L}^{-1}\{3G\} = \mathcal{L}^{-1}\left\{\frac{s-1+4}{(s-1)^2} - \frac{1}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)} + \frac{4}{(s-1)^2} - \frac{1}{s}\right\}$

$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2} - G\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - y$

$= te^t - y = te^t -$ this thing!

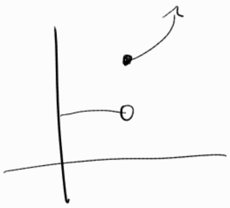
$y = \frac{1}{3}(4te^t + e^t - 1$

$\Uparrow$

$e^t + 4te^t - 1$

$\parallel$

$$F(t) = \begin{cases} 1 & t \in [0,1) \\ t^2+1 & t \geq 1 \end{cases}$$



$$= \mathcal{U}(t-1)(t^2+1) + (1-\mathcal{U}(t-1))(1)$$

$$x'' + w^2 x = F(t)$$