CALCULUS II: TRIGONOMETRIC INTEGRAL EXAMPLE

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Example 1. Solve

$$\int \tan^2 x \sec x \mathrm{d}x.$$

Solution. Normally, integrals involving tangent and secant terms multiplied together (specifically some even power of $\sec x$) are nicely solved with the substitution $u = \tan x$. However, since the power on secant is 1, we will need to try something else altogether.

Begin by making the Pythagorean substitution of $\sec^2 x - 1$ for $\tan^2 x$:

$$\int \left(\sec^2 x + 1\right) \sec x \mathrm{d}x = \int \sec^3 x - \sec x \mathrm{d}x \tag{1}$$

$$= \int \sec^3 x \mathrm{d}x - \int \sec x \mathrm{d}x \tag{2}$$

Splitting the integral up linearly, we first consider only the integral of $\sec x$. We can keep the integral unchanged by multiplying by 1, and we do so as follows:

$$\int \sec x \mathrm{d}x = \int \sec x \cdot \frac{\tan x + \sec x}{\tan x + \sec x} \mathrm{d}x \tag{3}$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x} \mathrm{d}x \tag{4}$$

substituting $u = \tan x + \sec x$, we get $\mathrm{d}u = (\sec x \tan x + \sec^2 x)\mathrm{d}x$:

$$= \int \frac{\mathrm{d}u}{u} = \ln|\tan x + \sec x| + C. \tag{5}$$

We now aim to solve the first integral in Equation 2: Using integration by parts, we have that

$$\int \sec^3 x \mathrm{d}x = \int (\sec x)(\sec^2 x) \mathrm{d}x \tag{6}$$

$$= \sec x \tan x - \int \tan^2 x \sec x \mathrm{d}x \tag{7}$$

and yet, the remaining integral is what we aimed to solve in the first place! Label this integral as I, then

$$I = \sec x \tan x + \ln|\tan x + \sec x| + C - I \tag{8}$$

and so

$$\int \tan^2 x \sec x \mathrm{d}x = \frac{1}{2}\left(\sec x \tan x + \ln|\tan x + \sec x|\right) + C. \tag{9}$$