GEOMETRIC LIMITS OF SUMS OF ITERATED AND FIXED POLYNOMIALS

ALEX KAPIAMBA, SCOTT KASCHNER, AND ELEANOR WAISS

Abstract. For the family of maps $f_n(z) = (p(z))^n + q(z)$ for polynomials p and q we show that the geometric limit of Julia sets as $n \to \infty$ exists and explicitly describe this set. For the family of maps $g_n(z) = p^n(z) + q(z)$, we provide necessary conditions for the geometric limit of Julia sets to exist, explicitly describe this set, and provide hyperbolic and Siegel examples.

Let $f_n \colon \mathbb{C} \to \mathbb{C}$ be given by a degree n polynomial. For any map $f \colon \mathbb{C} \to \mathbb{C}$, the filled Julia set for f, denoted $K(f)$, is the set of points that remain bounded under iteration by f. The purpose of this study is to explore a variety of contexts in which the limit of $K(f_n)$ in the Hausdorff topology as $n \to \infty$ exists and describe this limiting set. We use the notation $\mathbb{D}_r = \{z \in \mathbb{C} \colon |z| < r\}$ for the disk centered at zero with radius $r > 0$ and $\mathbb{D} = \mathbb{D}_1$ for the unit disk.

A 2012 study by Boyd and Schulz [4] included a result for the family $f_n(z) = z^n + c$, for complex parameter c. Among many other things, they proved

Theorem (Boyd-Shulz, 2012)**.** *If $f_n(z) = z^n + c$, then under the Hausdorff metric,*

$$\text{for any } |c| < 1, \ \lim_{n\to\infty} K(f_n) = \overline{\mathbb{D}};$$
$$\text{for any } |c| > 1, \ \lim_{n\to\infty} K(f_n) = \partial\mathbb{D}.$$

It was shown in [9] that when $|c| = 1$, the limiting of $K(f_n)$ almost always fails to exist. In another study by Alves [1], it was shown that $f_{n,c}(z) = z^n + cz^k$ with fixed positive integer k, if $|c| < 1$, then the limit of $K(f_{n,c})$ as $n \to \infty$ is $\partial\mathbb{D}$.

For $f_n(z) = z^n + q(z)$, where q is a fixed degree d polynomial, the limiting behavior of $K(f_n)$ is more interesting. In [5], this limit is shown to exist and described explicitly for most hyperbolic polynomial maps.

Theorem 1.1 (Brame-K, 2020 [5])**.** *For $f_n(z) = z^n + q(z)$, where q is a hyperbolic polynomial map with no attracting periodic points on the unit circle,*

$$\lim_{n\to\infty} K(f_n) = \mathcal{K}_q(\mathbb{D}) \cup \bigcup_{j=0}^{\infty} S_j,$$

where

$$\begin{aligned} \mathcal{K}_q(\mathbb{D}) &= \{z \in \mathbb{C} \colon q^i(z) \in \mathbb{D} \text{ for all nonnegative } j\} \text{ and} \\ S_j &= \{z \in \mathbb{C} \colon q^j(z) \in \partial\mathbb{D} \text{ and } q^i(z) \in \mathbb{D} \text{ for all } 0 \le i < j\}. \end{aligned}$$

The limit in Theorem 1.1 is the unit circle along with the set of points whose orbits remain bounded in the unit disk or eventually map to the unit circle. Heuristically for large n, the dynamics of $z \mapsto z^n + q(z)$ are dominated by q inside the unit disk (where z^n is very small) and dominated by z^n outside the unit disk (where z^n blows up). The assumption that q is hyperbolic is required to ensure that the interior of the filled Julia set is comprised exclusively of attracting basins. For large n, these attracting basins for p are used to approximate basins of attraction of $z \mapsto z^n + q(z)$ inside $\mathbb{D}$; any compact set in the complement of $\overline{\mathbb{D}}$ is in the basin of infinity. On

Date: August 8, 2024.

the unit circle, the dynamics of $z \mapsto z^n + q(z)$ are neither governed by z^n nor q exclusively, so we require additionally that the attracting points for q not be on the unit circle. Motivated by this, we can define $\hat{f}\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$\hat{f}(z) = \begin{cases} q(z), & \text{if } z \in \mathbb{D}, \\ z^2, & \text{if } z \in \partial\mathbb{D}, \text{ and} \\ \infty, & \text{if } z \notin \overline{\mathbb{D}}. \end{cases}$$

Defining $K(\hat{f})$ as the set of points whose orbits by $\hat{f}$ are bounded, it follows immediately from the definition of $\hat{f}$ that $K(\hat{f}) = \mathcal{K}_q(\mathbb{D}) \cup \bigcup_{j=0}^{\infty} S_j$, so the conclusion of Theorem 1.1 can be stated as

$$\lim_{n\to\infty} K(f_n) = K(\hat{f}).$$

In fact, a more general statement is true:

Theorem 1.2 (Brame-K, 2020 [5])**.** *For any polynomial q,*

$$\partial K(\hat{f}) \subseteq \varliminf_{n\to\infty} K(f_n) \subseteq \varlimsup_{n\to\infty} K(f_n) \subseteq K(\hat{f}). \tag{1}$$

The role of the unit disk and circle is due in large part to the use of power maps z^n, whose filled Julia sets are all the closed unit disk. Here we present generalizations of the results of [5] by replacing z^n with the "power" of a polynomial in two distinct ways: for the family of maps given by

$$h_n(z) = (p(z))^n + q(z),$$

where p and q are polynomials, and for the family of maps given by

$$g_n(z) = p^n(z) + q(z),$$

where p^n is the composition of p with itself n times. We will assume throughout that the degree of both p and q is greater than or equal to two; we leave the linear cases to future study.

Defining $\hat{h}\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$\hat{h}(z) = \begin{cases} q(z), & \text{if } z \in p^{-1}(\mathbb{D}), \\ z^2, & \text{if } z \in \partial p^{-1}(\mathbb{D}), \text{ and} \\ \infty, & \text{if } z \notin p^{-1}(\overline{\mathbb{D}}), \end{cases}$$

we have

Theorem 1.3. *For any polynomials, p and q, let $h_n(z) = (p(z))^n + q(z)$. If q is hyperbolic and has no attracting periodic points on $\partial p^{-1}(\mathbb{D})$, then*

$$\lim_{n\to\infty} K(h_n) = K(\hat{h}) = \mathcal{K}_q(p^{-1}(\overline{\mathbb{D}})) \cup \bigcup_{j=0}^{\infty} \mathcal{Q}_j,$$

where

$$\begin{aligned} \mathcal{K}_q(p^{-1}(\mathbb{D})) &= \{q^j(z) \in p^{-1}(\mathbb{D}) \text{ for all nonnegative } j\} \text{ and} \\ \mathcal{Q}_j &= \{q^j(z) \in \partial p^{-1}(\mathbb{D}) \text{ and } q^k(z) \in p^{-1}(\mathbb{D}) \text{ for } k = 1, \ldots, j-1\}. \end{aligned}$$

This result is quite similar to that of [5], but the unit disk is replaced with its preimage by p. See Figures 1 and 2. More specifically, the family of maps h_n is the family of maps from [5], but the z^n term is replaced with $(p(z))^n$. As a result, the arguments presented in [5] can be adapted with minimal change to prove Theorem 1.3; we leave the details to the reader.

We denote the interior of $K(p)$ as $\operatorname{int} K(p)$. For the family of maps g_n, if we again assume that p is hyperbolic, then all points in $\operatorname{int} K(p)$ will be attracted to a periodic attracting k-cycle for p, which we denote $z_1, \ldots, z_k$. Thus, for each n there is some indexing of the periodic cycle, say $z_{\ell_1}, \ldots, z_{\ell_k}$, such that we can approximate $g_n(z)$ with a piecewise function whose pieces are given

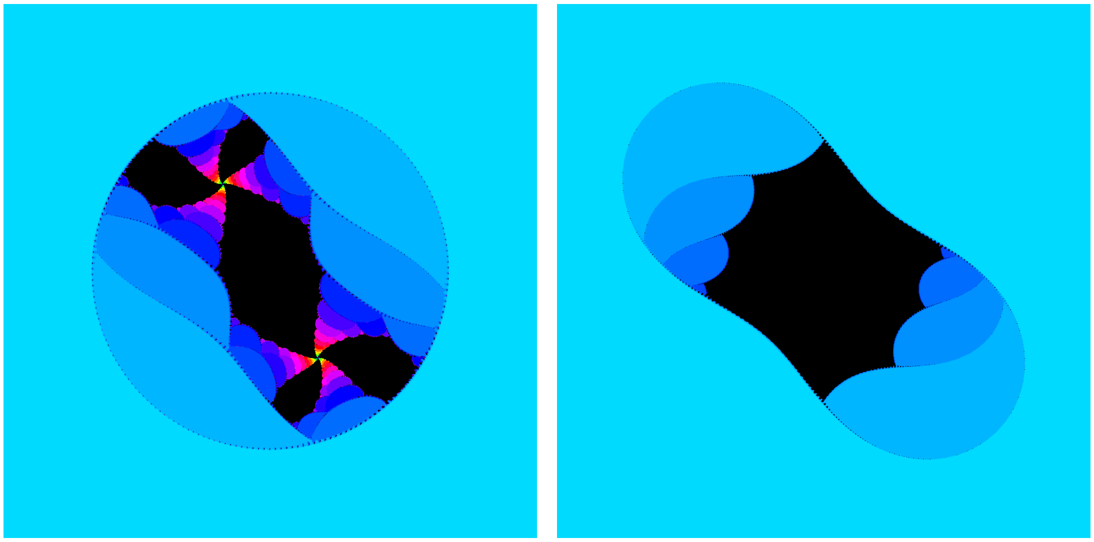

FIGURE 1. On the left is $K(z^{200} + q(z))$, where $q(z) = z^2 - 0.123 + 0.745i$; on the right is $K(f_{200})$, where $p(z) = z^2 - 0.123 + 0.745i$ and $q(z) = z^2 + 0.12 - 0.3i$. The color gradation in the images indicates the number of iterates required to exceed a fixed bound for modulus; points colored black do not reach this bound in a fixed number of iterates.

by $q(z) + z_{\ell_i}$ for $i = 1, \ldots, k$; this approximation gets better with larger n. Note then that, in particular, unless p has an attracting fixed point in $\mathbb{C}$ (that is, $k = 1$), $\lim_{n\to\infty} K(g_n)$ cannot exist. However, these limits can certainly exist along subsequences, see Figure 3.

Definition 1.4. *A subsequence n_m is p-*convergent *if p^{n_m} converges on $\operatorname{int} K(p)$, that is, if for each $z \in \operatorname{int} K(p)$, $\lim_{m\to\infty} p^{n_m}(z)$ exists.*

Proposition 1.5. *If p is hyperbolic and k is the least common multiple of the lengths of its attracting cycles, then each subsequence $n_m = \ell + mk$ for $\ell \in \{1, \ldots, k\}$ is p-convergent.*

For a p-convergent sequence n_m, define $\hat{g}\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$\hat{g}(z) = \begin{cases} q(z) + \lim_{m\to\infty} p^{n_m}(z), & \text{if } z \in \operatorname{int} K(p), \\ p(z), & \text{if } z \in J(p), \text{ and} \\ \infty, & \text{if } z \notin K(p), \end{cases}$$

Theorem 1.6. *For any polynomials p and q and p-convergent sequence n_m for p,*

$$\partial K(\hat{g}) \subseteq \varliminf_{m\to\infty} \partial K(g_{n_m}) \subseteq \varlimsup_{m\to\infty} K(g_{n_m}) \subseteq K(\hat{g}). \tag{2}$$

Theorem 1.7. *Let p and q be polynomials such that n_m is p-convergent, $\hat{g}$ has no attracting periodic points on $J(p)$, and $\hat{g}$ has only topologically attracting basins in $\operatorname{int} K(\hat{g})$. Then*

$$\lim_{m\to\infty} K(g_{n_m}) = K(\hat{g}) = \mathcal{K}_{\hat{g}}(\operatorname{int} K(p)) \cup \bigcup_{j=0}^{\infty} \mathcal{J}_j.,$$

where

$$\begin{aligned} \mathcal{K}_{\hat{g}}(\operatorname{int} K(p)) &= \{\hat{g}^j(z) \in \operatorname{int} K(p) \text{ for all nonnegative } j\} \text{ and} \\ \mathcal{J}_j &= \{\hat{g}^j(z) \in \partial K(p) \text{ and } \hat{g}^k(z) \in \operatorname{int} K(p) \text{ for } k = 1, \ldots, j-1\}. \end{aligned}$$

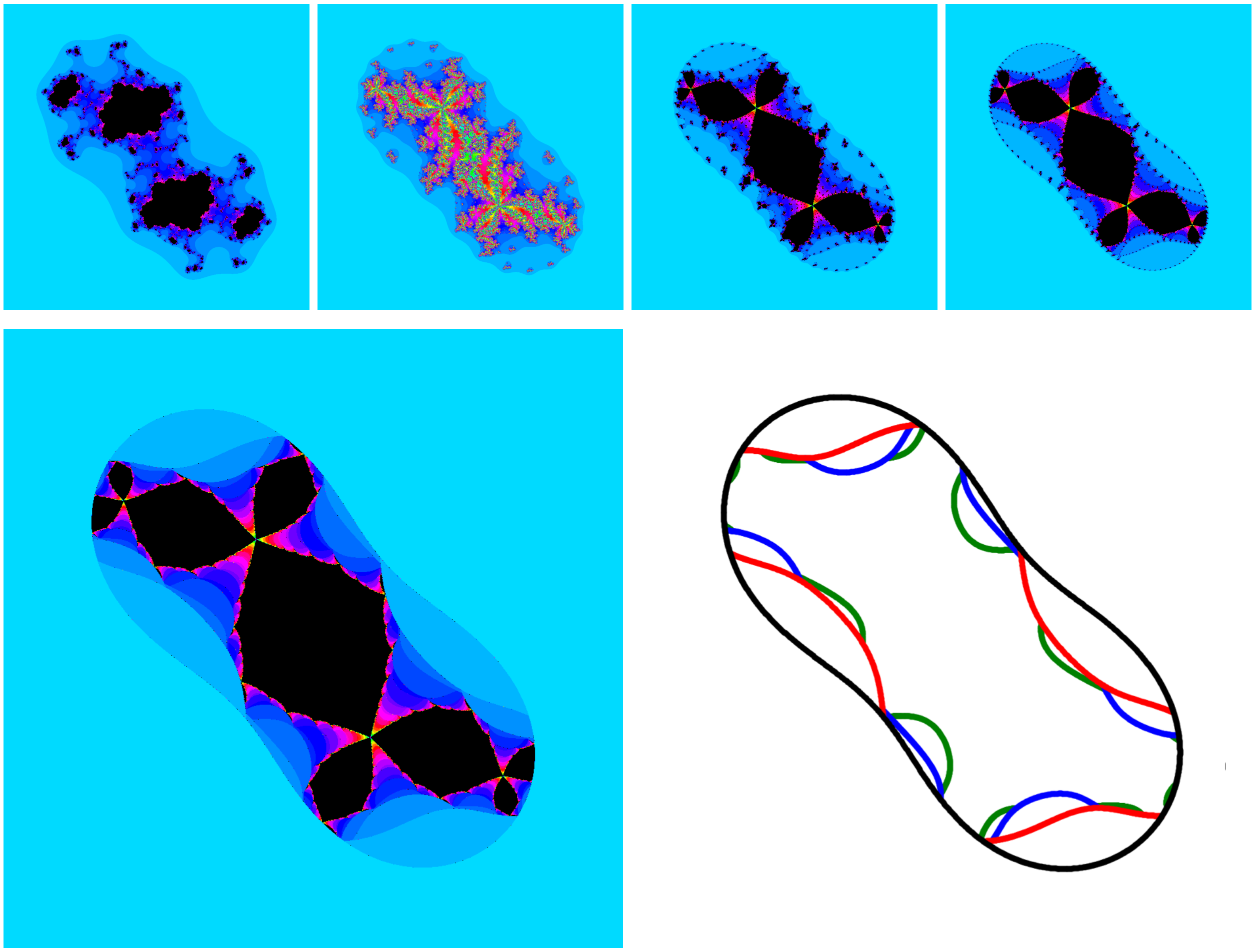

FIGURE 2. Top: $K(f_n)$ for $p(z) = z^2 + 0.05 + 0.745i$, $q(z) + z^2 - 0.123 + 0.745i$, $n = 6, 12, 25, 50$. Bottom left: $K(f_{1800})$ for the same p and q. The color gradation in the images indicates the number of iterates required to exceed a fixed bound for modulus; points colored black do not reach this bound in a fixed number of iterates. Bottom right: (black) $p^{-1}(\mathbb{D})$ and $\mathcal{Q}_j$ for $j = 1, 2, 3$ (red, blue, green).

This result is similar to that of [5], but the role of the unit disk is played by the filled Julia set for p. See Figure 4. In fact, for $p(z) = z^2$, all subsequences of $\mathbb{N}$ are p-convergent, $\hat{g} = q$ on $\operatorname{int} K(p) = \mathbb{D}$, and Theorems 1.1 and 1.2 from [5] follow from Theorem 1.7.

In Section 2, we provide a quick tour of background information, in Section 3, we discuss and prove results for the family g_n, and in Section 4 we provide examples and counterexamples for limiting behavior of $K(g_n)$.

2. Background, Notation and Terminology

2.1. Notation and Terminology. Given two sets A, B in a metric space (X, d), the Hausdorff distance $d_{\mathcal{H}}(A, B)$ between the sets is defined as

$$d_{\mathcal{H}}(A, B) \quad = \quad \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}.$$

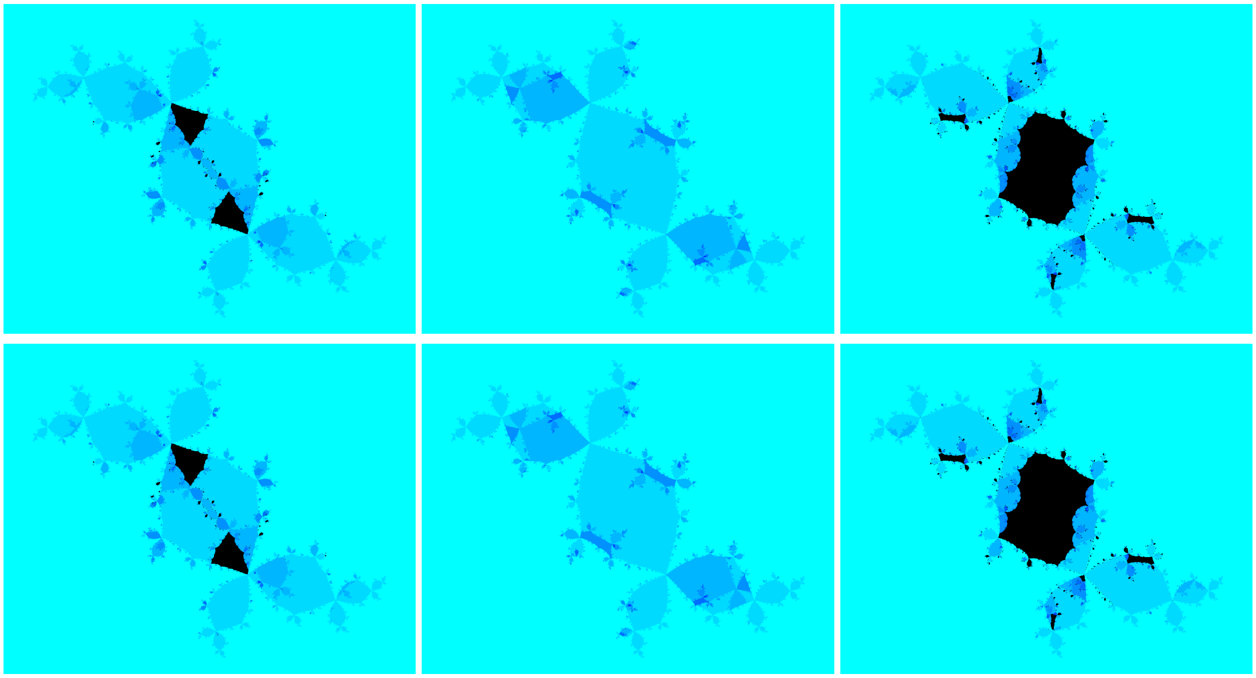

FIGURE 3. $K(g_n)$ for $p(z) = z^2 - 0.123 + 0.745i$ and $q(z) = z^2 + 0.12 - 0.3i$. From left to right, (top) $n = 49$, 50, and 51; (bottom) $n = 52$, 53, and 54. The color gradation in the images indicates the number of iterates required to exceed a fixed bound for modulus; points colored black do not reach this bound in a fixed number of iterates.

Each point in A has some minimal distance to B, and vice versa. The Hausdorff distance is the maximum of all these distances.

Suppose S_n and S are compact subsets of $\mathbb{C}$. If for all $\varepsilon > 0$, there is $N > 0$ such that for any $n \geq N$, we have $d_{\mathcal{H}}(S_n, S) < \varepsilon$, then we say S_n converges to S and write $\lim_{n\to\infty} S_n = S$.

We also use of Painlevé-Kuratowski set convergence [11]. For a sequence of sets, S_n, we have

$$\begin{aligned} \varliminf_{n\to\infty} S_n &= \{z \in \mathbb{C} \colon \varlimsup_{n\to\infty} d(z, S_n) = 0\}, \\ \varlimsup_{n\to\infty} S_n &= \{z \in \mathbb{C} \colon \varliminf_{n\to\infty} d(z, S_n) = 0\}. \end{aligned}$$

It follows immediately that $\varliminf_{n\to\infty} S_n \subset \varlimsup_{n\to\infty} S_n$. We say S_n converges to a set S in the sense of Painlevé-Kuratowski if $\varliminf_{n\to\infty} S_n = \varlimsup_{n\to\infty} S_n = S$, or equivalently, $\varlimsup_{n\to\infty} S_n \subseteq \varliminf_{n\to\infty} S_n = S$. In [7], it is shown that for sequences of bounded sets, Painlevé-Kuratowski set convergence agrees with convergence with Hausdorff distance.

2.2. **Complex Dynamics.** Let us briefly review some classical definitions and facts in complex dynamics. More details and proofs of these statements can be found in [3, 6, 10].

For a rational map, f, from $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to $\hat{\mathbb{C}}$, the Fatou set, denoted $\mathcal{F}(f)$, is the set of points for which the iterates of f form a normal family; the Julia set of f, denoted $J(f)$, is the complement of $\mathcal{F}(f)$ in $\hat{\mathbb{C}}$. The set of points whose orbits remain bounded the filled Julia set, denoted $K(f)$ as above. Julia sets $J(f)$ and filled Julia sets $K(f)$ are compact in $\hat{\mathbb{C}}$. When f is a polynomial map, $J(f)$ is the boundary of $K(f)$.A point $z \in \hat{\mathbb{C}}$ is exceptional for f if $f^{-1}(z) = \{z\}$. The set $\mathcal{E}(f)$ of all exceptional points for f contains at most two points, one of those points is infinity when f is a

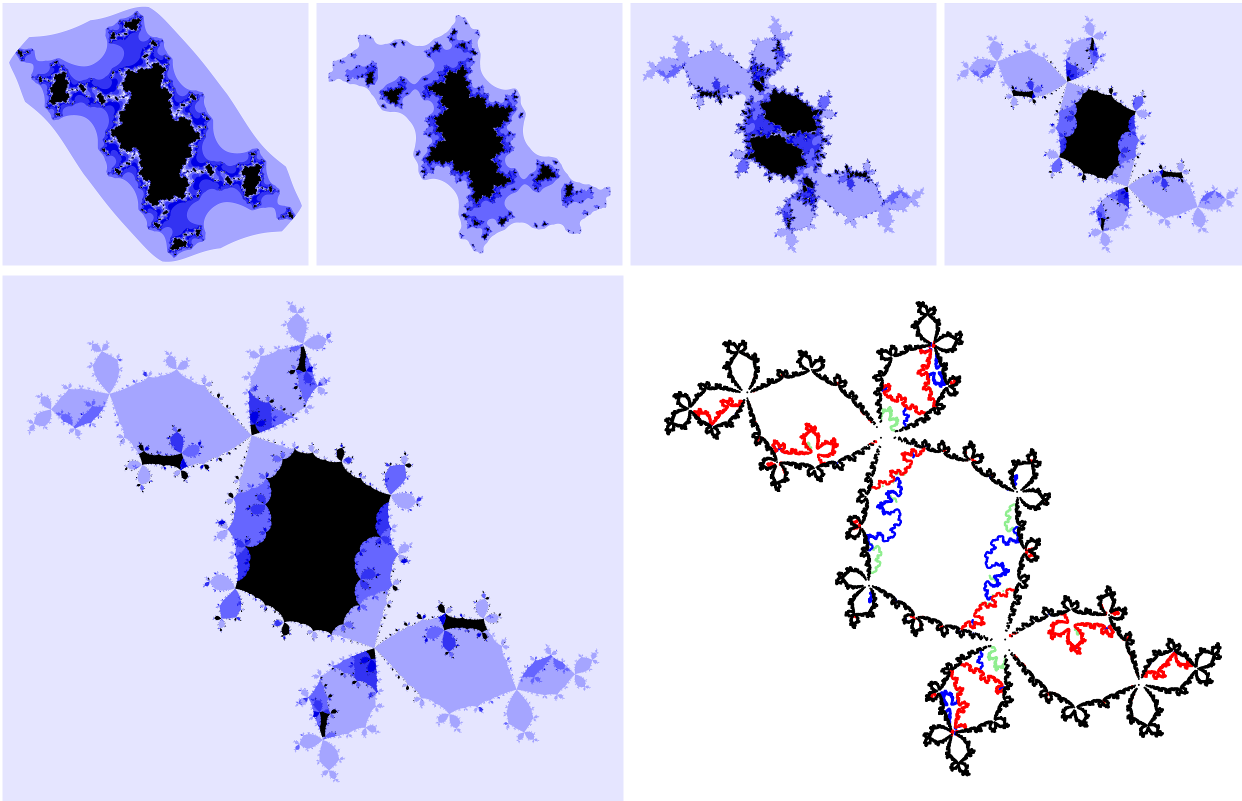

FIGURE 4. Top: $K(g_n)$ for $p(z) = z^2 - 0.123 + 0.745i,\ q(z) + z^2 + 0.12 - 0.3i$, $n_m = 3, 6, 15, 150$ (i.e. along the p-convergent subsequence 0 mod 3). The color gradation in the images indicates the number of iterates required to exceed a fixed bound for modulus; points colored black do not reach this bound in a fixed number of iterates. Bottom left: $K(g_{1500})$ for the same p and q. Bottom right: $J(p)$ (black) and $\mathcal{J}_j$ for $j = 1, 2, 3$ (red, blue, green).

polynomial. For any open set U which intersects the Julia set of f and any $z \in \hat{\mathbb{C}} \setminus \mathcal{E}(f)$, there is an integer $n \geq 1$ such that $z \in f^n(U)$.

A point $z \in \hat{\mathbb{C}}$ is periodic for f with period k if $f^k(z) = z$ and $z, f(z), \ldots, f^{k-1}(z)$ are all distinct. The multiplier λ of a periodic point z_0 of period k is $\lambda = (f^k)'(z_0)$. If $|\lambda| < 1$, z_0 is said to be attracting; if $\lambda > 1$, z_0 is repelling; if $\lambda = 1$, z_0 is indifferent. All repelling periodic points are contained in $J(f)$, and repelling periodic points are dense in $J(f)$. Attracting periodic points are contained in $\mathcal{F}(f)$. Moreover, for every attracting periodic point z_0 of period k, there is an open neighborhood U of z_0 such that $f^k(U) \subset U$ and the orbit by f^k of any point in U converges to z_0. The set of all points whose orbits by f^k converge to z_0 is the basin of attraction for z_0. More generally, we say a periodic point of period k, z_0, of a map f (not necessarily rational) is topologically attracting if there is a neighborhood U of z_0 such that $\{f^{nk}|_U\}$ converges uniformly to the constant map $U \to z_0$; a basin containing a topologically attracting periodic point is called a topologically attracting basin. Finally, a map is hyperbolic if there is a conformal metric μ defined in an open neighborhood of of $J(f)$ at every point $z \in J(f)$, we have $\|Df_z(v)\|_\mu > \|v\|_\mu$ for every nonzero v in the tangent space $T\hat{\mathbb{C}}_z$. An equivalent definition of hyperbolicity (for rational maps) is a map is hyperbolic if every point in $\mathcal{F}(f)$ converges to an attracting periodic cycle.

3. Iterates of Polynomials

In [2], Bayraktar and Efe show that $\mathcal{J}_0 = J(p) \subset \liminf_{n\to\infty} K(g_n)$, and their techniques apply to a much wider class of polynomial function. However, since our sequence of maps, g_n, is changing in a dynamical sense (iteration rather than powers), some aspects of the proof of Theorem 1.7 are much simpler, so we include them here. For example, if for $K \subset \mathbb{C}$, we define

$$\begin{aligned} K_{-\varepsilon} &= \{z \in K\colon d(z, \partial K) \geq \varepsilon\}, \text{ and} \\ K_{+\varepsilon} &= \{z \in \mathbb{C}\colon d(z, K) \leq \varepsilon\}, \end{aligned}$$

then proof of the following lemma is nearly trivial.

Lemma 3.1. *For all $\varepsilon > 0$, there is an N such that for any $n \geq N$,*

$$K(g_n) \subseteq K(p)_{+\varepsilon}.$$

Proof. Let $R > 0$ be large enough so that if $z \notin \mathbb{D}_R$ then $|p(z)| > 3|z|$. Let $N \geq 1$ be large enough so that $|q(z)| < |z|^n$ for all $z \in \mathbb{C}$ and $n \geq N$. For any $z \notin \mathbb{D}_R$ and $n \geq N$, we have

$$|g_n(z)| \geq |p^n(z)| - |q(z)| \geq (3^n - 1)|z|^n \geq 2|z|,$$

so $\mathbb{C} \setminus \mathbb{D}_R$ is in the basin of infinity for g_n.

For any $\varepsilon > 0$, we can choose $R > 0$ above large enough so that $K(p)_{+\varepsilon} \subset \mathbb{D}_R$. Setting $A = \max_{z \in \overline{\mathbb{D}_R}} q(z)$, there is an $N \geq 1$ such that for any $z \in \mathbb{D}_R \setminus K(p)_{+\varepsilon}$ and all $n \geq N$ we have $p^n(z) \notin \mathbb{D}_{R+A}$, hence $g_n(z) \notin \mathbb{D}_R$. Thus $\mathbb{C} \setminus K(p)_{+\varepsilon}$ is in the basin of infinity for g_n. $\square$

Lemma 3.2. *If n_m is p-convergent, then*

$$\bigcup_{j=0}^{\infty} \mathcal{J}_j \subset \varliminf_{n\to\infty} \partial K(g_{n_m}).$$

Proof. We first show that $\mathcal{J}_0 = J(p) \subset \varliminf_{n\to\infty} \partial K(g_n)$. Let $z_0 \in J(p)$ be a repelling periodic point of period $k \geq 1$ such that $-q(z_0) \notin \mathcal{E}(p)$. Let us observe that if ε is sufficiently small, then there exists topological disks V_n for $n \geq 0$ such that $\lim_{n\to\infty} \overline{V_n} = \{z_0\}$ and p^n maps V_n to V as a covering map whose only possible branch value is $-q(z_0)$. Indeed, as $-q(z_0)$ is not exceptional for p we can pull back V by iterates of p to get close to z_0, and as z_0 is a repelling periodic point we can then pull back by the local branch of p^{-k} which fixes z_0 to converge towards z_0. Let U be the disk of radius ε centered at z_0. For large $n \geq 0$, the map

$$f(z) = p^n(z) + q(z_0)$$

maps $V_n \subset U$ to U_0, so f has a fixed point in V_n. When n is sufficiently large, so in particular $\overline{V_n}$ is close to z_0, we have

$$|f(z) - z| > \frac{\varepsilon_0}{2},$$

and

$$|f(z) - g_n(z)| = |q(z_0) - q(z)| < \frac{\varepsilon_0}{2}$$

for all $z \in \partial V_n$. It therefore follows from Rouché's theorem that g_n has a fixed point in $V_n \subset U$. Thus for all large n there exists a point $z \in K(g_n)$ with $|z - z_0| < \varepsilon$. As $z \in \partial K(p)$, Lemma 3.1 implies that for all large n there is also a point $z \notin K(g_n)$ with $|z - z_0| < \varepsilon$. Hence there must be some point $z \in J(g_n)$ with $|z - z_0| < \varepsilon$. As ε can be chose arbitrarily small, it follows that $z_0 \in \varliminf_{n\to\infty} \partial K(g_n)$. As the repelling periodic points of p are dense in $J(p)$ and $\mathcal{E}(p)$ contains at most one point in $\mathbb{C}$, it follows that $J(p) \subset \varliminf_{n\to\infty} \partial K(g_n)$.

The result now follows from backward invariance of $J(g_{n_m})$ and the fact that for any compact subset, K, of the interior of $K(p)$ and all $\varepsilon > 0$, there is an N such that

$$d(g_{n_m}(z), \hat{g}(z)) < \varepsilon$$

for all $z \in K$ any $m \geq N$. $\square$

Proof of Theorem 1.6. By construction, when there are an infinite number of nonempty $\mathcal{J}_j$, they accumulate on $\partial \bigcap_{j=0}^{\infty} \hat{g}^{-j}(\operatorname{int} K(p))$. Using this with Lemma 3.2, we have

$$\left(\partial \bigcap_{j=0}^{\infty} \hat{g}^{-j}(\operatorname{int} K(p))\right) \cup \bigcup_{j=0}^{\infty} \mathcal{J}_j = \partial K(\hat{g}) \subseteq \varliminf_{m\to\infty} \partial K(g_{n_m}).$$

Next, we have that $\varliminf_{m\to\infty} \partial K(g_{n_m}) \subset \varlimsup_{m\to\infty} K(g_{n_m})$, which follows from definition. Lastly, note that if $z \notin K(p)_{+\varepsilon}$, then by Lemma 3.1, we know $z \notin \varlimsup_{m\to\infty} K(g_{n_m})$. Moreover, if $z \in K(p)_{+\varepsilon}$, then we may assume $z \in K(p)_{-\varepsilon} \setminus \bigcup_{j=0}^{\infty} \mathcal{J}_j$ because $\bigcup_{j=0}^{\infty} \mathcal{J}_j$ is already in $\varlimsup_{m\to\infty} K(g_{n_m})$ by Lemma 3.2. Then z is either in $\bigcap_{j=0}^{\infty} \hat{g}^{-j}(\operatorname{int} K(p))$ or z maps by g_n outside $K(p)_{+\varepsilon}$ for all sufficiently large n. Thus,

$$\varlimsup_{m\to\infty} K(g_{n_m}) \subseteq \bigcap_{j=0}^{\infty} \hat{g}^{-j}(\operatorname{int} K(p)) \cup \bigcup_{j=0}^{\infty} \mathcal{J}_j = K(\hat{g}).$$

$\square$

Proof of Theorem 1.7. Theorem 1.6 showed $\partial K(\hat{g}) \subseteq \varliminf_{m\to\infty} K(g_{n_m})$. Hence, it remains only to show that

$$\operatorname{int}\left(\bigcap_{j=0}^{\infty} \hat{g}^{-j}(\operatorname{int} K(p))\right) \subset \varliminf_{m\to\infty} K(g_{n_m}).$$

Let $K = \bigcap_{j=0}^{\infty} \hat{g}^{-j}(\operatorname{int} K(p))$, so $\overline{K}_{-\varepsilon}$ is a compact subset of K whose elements at least ε away from the boundary of K. By assumption, $\hat{g}$ has only topologically attracting components in $\operatorname{int} K(\hat{g})$, hence we have that $\hat{g}^k(\overline{K}_{-\varepsilon})$ converges uniformly to some subset of the attracting periodic points of $\hat{g}$ as $k \to \infty$ in the Hausdorff sense. We denote this limit as $B = \lim_{k\to\infty} \hat{g}^k(\overline{K}_{-\varepsilon})$ and note that B is comprised of attracting periodic points for $\hat{g}$. Because we assume $\hat{g}$ has no attracting periodic points on $J(p)$, for sufficiently small $\varepsilon_0 > 0$ (and possibly smaller ε), we have $B_{+\varepsilon_0} \subset \overline{K}_{-\varepsilon}$.

Let $k_0 \geq 0$ be large enough so that $\hat{g}^{k_0}(\overline{K}_{-\varepsilon}) \subset \operatorname{int} B_{+\varepsilon_0}$. By assumption g_{n_m} converges uniformly to $\hat{g}$ on $\overline{K}_{-\varepsilon}$, thus if m is sufficiently large then $g_{n_m}^{k_0}(\overline{K}_{-\varepsilon}) \subset B_{+\varepsilon_0} \subset \overline{K}_{-\varepsilon}$. Hence $\overline{K}_{-\varepsilon} \subset K(g_{n_m})$ for all large m, so $\overline{K}_{-\varepsilon} \subset \varliminf_{m\to\infty} K(g_{n_m})$. Taking $\varepsilon \to 0$ completes the proof. $\square$

4. Examples

Throughout this section, the color gradation in images indicates the number of iterates required to exceed a fixed bound for modulus; points colored black do not reach this bound in a fixed number of iterates.

Example 1. Let $p(z) = z^2 + c$ and $q(z) = z^2 + d$, where c is chosen so that is p has an attracting fixed point, z_0. Since p has an attracting fixed point, any subsequence of positive integers is p-convergent, and we have

$$\hat{g}(z) = q(z) + z_0$$

Now choose d so that $\hat{g}$ has an attracting cycle $\{z_1, \ldots, z_k\} \subset \operatorname{int} K(p)$. In that case, we also have $\{z_1, \ldots, z_k\} \subset \bigcap_{j=0}^{\infty} \hat{g}^{-j}(\operatorname{int} K(p))$. Thus, we may apply Theorem 1.7.

More specifically, when $c = 0.7$, we have $z_0 \approx -0.47468$, so choosing $d = 0.4i$, we have $q(z) \approx z^2 - 0.47468 + 0.4i$, which is hyperbolic with an attracting fixed point $z_1 \approx -0.38103 + 0.227i \in K(p)$. It follows that $\lim_{k\to\infty} q^k(\operatorname{int} K(p)) = \{z_1\} \in \operatorname{int} K(p)$; see Figure 5.

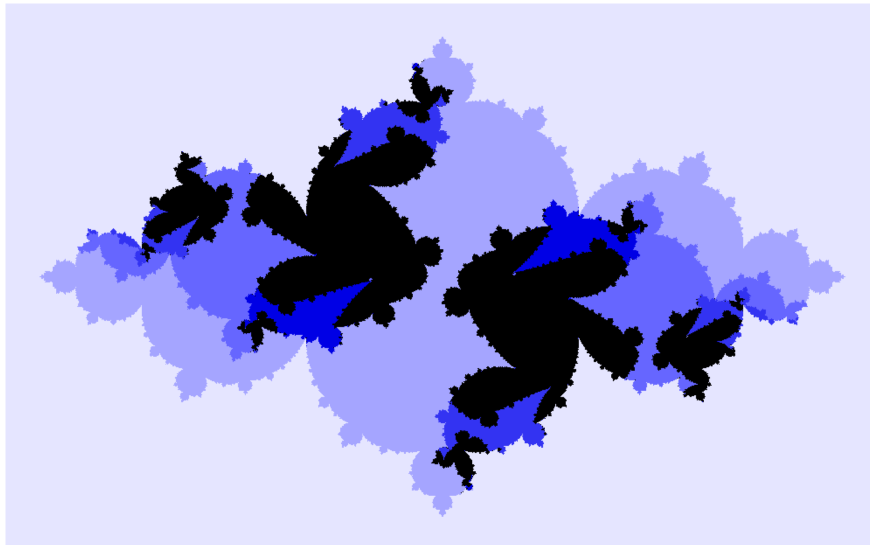

FIGURE 5. $\lim_{n\to\infty} K(g_n)$ when $p(z) = z^2 - 0.7$ and $q(z) = z^2 + 0.4i$.

Example 2. When p has an attracting k-cycle, for any $k \geq 2$, $\hat{g}$ is a piecewise polynomial, and verifying the hypotheses of Theorem 1.7 is more complicated. Let $p(z) = z^2 - 1$ and $q(z) = \alpha z^2 + 0.35 - 0.2i$, so p has an attracting 2-cycle, $\{0, -1\}$, and for $|\alpha| < 1/100$, q has an attracting fixed point at $z_0 \approx 0.35 + 0.2i \in \operatorname{int} K(p)$. Since p has an attracting two-cycle, any subsequence of odd positive integers is p-convergent, and any subsequence of even positive integers is p-convergent. Working with the p-convergent subsequence of even integers, we have

$$\hat{g}(z) = \begin{cases} q(z), & \text{if } z \text{ is in the basin of } 0 \text{ for } p^2; \\ q(z) - 1, & \text{if } z \text{ is in the basin of } -1 \text{ for } p^2. \end{cases}$$

Since z_0 is in the basin of 0 for p, where $\hat{g} = q$, we have $z_0 \in \bigcap_{j=0}^{\infty} \hat{g}^{-j}(\operatorname{int} K(p))$. Let K be a compact subset of $\bigcap_{j=0}^{\infty} \hat{g}^{-j}(\operatorname{int} K(p))$. Note first that for any $z \in K$, we have $q(z) \in \{z \in \mathbb{C}\colon |z - z_0| < 7/100\} \subset \operatorname{int} K(p)$ and $q(z) + 1 \in \{z \in \mathbb{C}\colon |z - z_0 + 1| < 7/100\} \subset \mathbb{C} \backslash K(p)$. It follows that if z is the basin of -1 for p, then $\hat{g}(z) \notin \operatorname{int} K(p)$. Moreover, if z is the basin of 0 for p, then $\hat{g}(z)$ is in the immediate basin of 0 for p. Thus, we may apply Theorem 1.7; see Figure 6.

Example 3. Let $p(z) = e^{2\pi i\theta} z + z^2$, where $\theta = (\sqrt{5} + 1)/2$ is the golden mean. The map p has a Siegel disk, Δ, centered at $z = 0$, so $p|_{\Delta}$ is conjugate by an invertible map ϕ to a map $R\colon \mathbb{D}_r \to \mathbb{D}_r$ given by $R(z) = e^{2\pi i\theta} z$. Since θ is of bounded type, the critical point for p is on the boundary of the Siegel disk, in $J(p)$. Thus, the interior of $K(p)$ is comprised of just the Siegel disk and its preimages by p.

The Fibonacci sequence, F_n, satisfies $\lim_{n\to\infty} F_{n+1}/F_n = \theta$, so R^{F_n} converges to the identity map as $n \to \infty$. It follows that F_n is p-convergent because for any z in the interior of the Siegel disk,

$$\lim_{n\to\infty} p^{F_n}(z) = (\phi^{-1} \circ R^{F_n} \circ \phi)(z) = z.$$

Then $\hat{g}(z) = z + q(z)$, so any choice of q for which $\hat{g}$ has only attracting Fatou basins satisfies the hypotheses of Theorem 1.7. For example, when $q(z) = z^2 - 0.2 - 0.4i$, $\hat{g}$ has an attracting fixed point at $z_0 \approx -0.57 - 0.35i$; see Figure 7.

Example 4. Let $p(z) = z^2$ and $q(z) = 2z^2 + 1/8$ and note that in this example, all subsequences are p-convergent, and $K(q)$ is a strict subset of $K(p) = \overline{\mathbb{D}}$. In [8], Douady proved that $K(q)$ does

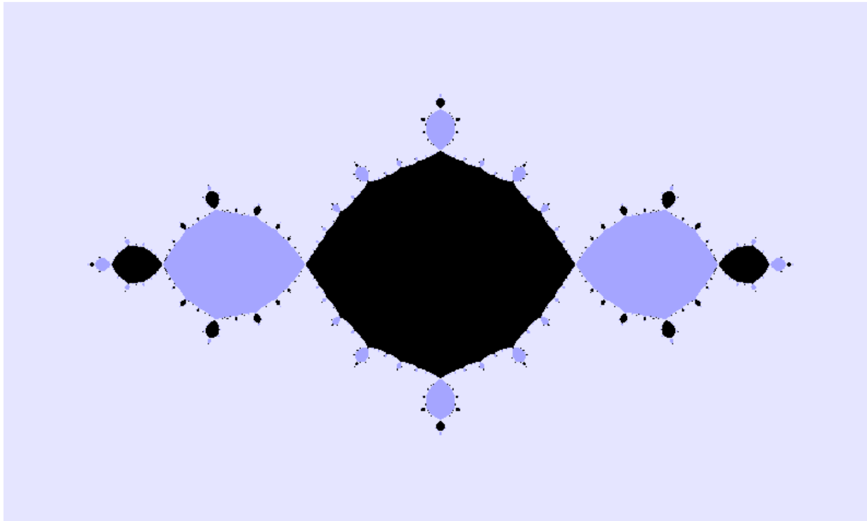

FIGURE 6. $\lim_{n\to\infty} K(g_n)$ through even n when $p(z) = z^2 - 1$ and $q(z) = \frac{1}{100}z^2 + 0.35 - 0.2i$.

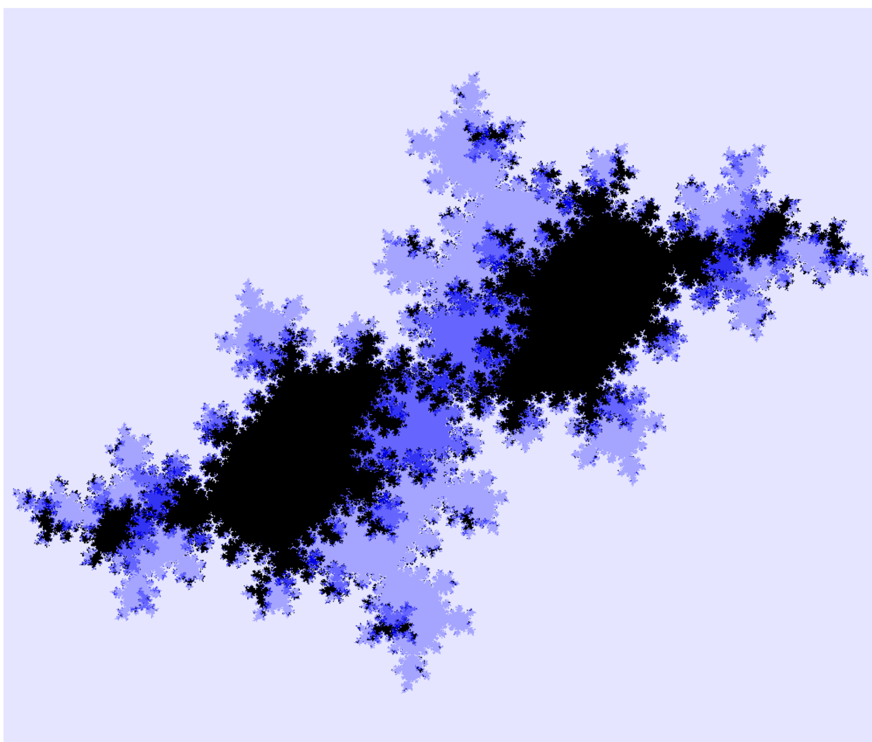

FIGURE 7. Approximation of $\lim_{n_m\to\infty} K(g_{n_m})$ through the Fibonacci sequence when $p(z) = e^{2\pi i\theta}z + z^2$, $\theta = (\sqrt{5}+1)/2$, and $q(z) = z^2 - 0.2 - 0.4i$.

not depend continuously on q; in particular, there is a sequence of positive numbers, ε_n, such that $\varepsilon_n \to 0$ and if $q_n(z) = q(z) + \varepsilon_n$, then the limit of $K(q_n)$ as $n \to \infty$ exists and is a strict subset of $K(q)$. Since g_n converges to q uniformly on compact subsets of $\mathbb{D}$, one can use similar techniques to show that there is a subsequence n_m so that limit of $K(g_{n_m})$ as $m \to \infty$ exists and is a strict subset of $K(\hat{g})$. The primary complication is defining appropriate Fatou coordinates for each g_n, but these are guaranteed to exist by the work of Shishikura in [12]. Note that Theorem 1.7 does not apply to this example because int $K(\hat{g})$ has a basin that is not attracting.

References

[1] Alexandre Miranda Alves. Geometric limits of julia sets and connectedness locus of the family of polynomials $p_c(z) = z^n + cz^k$. *Dynamical Systems*, (just-accepted):1–17, 2019.

[2] Turgay Bayraktar and Melike Efe. On dynamics of asymptotically minimal polynomials. *Journal of Approximation Theory*, 295:105956, 2023.

[3] Alan F. Beardon. *Iteration of rational functions*, volume 132 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Complex analytic dynamical systems.

[4] Suzanne Hruska Boyd and Michael J. Schulz. Geometric limits of Mandelbrot and Julia sets under degree growth. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 22(12):1250301, 21, 2012.

[5] Micah Brame and Scott Kaschner. Geometric limits of julia sets for sums of power maps and polynomials. *Complex Analysis and its Synergies*, 6:1–8, 2020.

[6] Lennart Carleson and Theodore W Gamelin. *Complex dynamics*. Springer Science & Business Media, 2013.

[7] Asen L Dontchev and R Tyrrell Rockafellar. Implicit functions and solution mappings. *Springer Monographs in Mathematics. Springer*, 208, 2009.

[8] Adrien Douady. Does a julia set depend continuously on the polynomial. *Complex dynamical systems: The mathematics behind the Mandelbrot set and Julia sets*, 49:91–138, 1994.

[9] Scott R Kaschner, Reaper Romero, and David Simmons. Geometric limits of julia sets of maps $z^n + \exp(2\pi i\theta)$ as $n \to \infty$. *International Journal of Bifurcation and Chaos*, 25(08):1530021, 2015.

[10] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.

[11] R Tyrrell Rockafellar and Roger J-B Wets. *Variational analysis*, volume 317. Springer Science & Business Media, 2009.

[12] Mitsuhiro Shishikura. Bifurcation of parabolic fixed points. *London Mathematical Society Lecture Note Series*, 1(274):325–364, 2000.

Email address: kapiamba@math.harvard.edu

Harvard University, Department of Mathematics, 1 Oxford St., Cambridge, MA 02138, United States

Email address: skaschne@butler.edu

Butler University, Mathematical Sciences, JH270, 4600 Sunset Ave., Indianapolis, IN 46208, United States

Email address: ewaiss@butler.edu

Butler University, Mathematical Sciences, JH270, 4600 Sunset Ave., Indianapolis, IN 46208, United States